THEORY OF EQUATIONS

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THE THEORY OF EQUATIONS:

WITH AN

INTRODUCTION TO THE THEORY OF BINARY ALGEBRAIC FORMS.

BY

WILLIAM SNOW BURNSIDE, M.A., D.Sc.,
FELLOW OF TRINITY COLLEGE, DUBLIN;
ERASMUS SMITH'S PROFESSOR OF MATHEMATICS IN
THE UNIVERSITY OF DUBLIN:

AND

ARTHUR WILLIAM PANTON, M.A., D.Sc.,
FELLOW AND TUTOR, TRINITY COLLEGE, DUBLIN;
DONEGAL LECTURER IN MATHEMATICS.

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PREFACE.

We have endeavoured in the present work to combine some of the modern developments of Higher Algebra with the subjects usually included in works on the Theory of Equations. The first ten Chapters contain all the propositions ordinarily found in elementary treatises on the subject. In these Chapters we have not hesitated to employ the more modern notation wherever it appeared that greater simplicity or comprehensiveness could be thereby obtained.

Regarding the algebraical and the numerical solution of equations as essentially distinct problems, we have purposely omitted in Chap. VI. numerical examples in illustration of the modes of solution there given of the cubic and biquadratic equations. Such examples do not render clearer the conception of an algebraical solution; and, for practical purposes, the algebraical formula may be regarded as almost useless in the case of equations of a degree higher than the second.

In the treatment of Elimination and Linear Transformation, as well as in the more advanced treatment of Symmetric Functions, a knowledge of Determinants is indispensable. We have found it necessary, therefore, to give a Chapter on this subject. It has been our aim to make this Chapter as simple and intelligible as possible to the beginner; and at the same time to omit

no proposition which might be found useful in the application of this calculus. For many of the examples in this Chapter, as well as in other parts of the work, we are indebted to the kindness of Mr. Cathcart, Fellow of Trinity College.

We have approached the consideration of Covariants and Invariants through the medium of the functions of the differences of the roots of equations—this appearing to us the simplest mode of presenting the subject to beginners. We have attempted at the same time to show how this mode of treatment may be brought into harmony with the more general problem of the linear transformation of algebraic forms. In the Chapters on this subject we have confined our attention to the quadratic, cubic, and quartic; regarding any complete discussion of the covariants and invariants of higher binary forms as too difficult for a work like the present.

Of the works which have afforded us assistance in the more elementary part of the subject, we wish to mention particularly the *Traité d' Algèbre* of M. Bertrand, and the writings of the late Professor Young* of Belfast, which have contributed so much to extend and simplify the analysis and solution of numerical equations.

In the more advanced portions of the subject we are indebted mainly, among published works, to the *Lessons Introductory to the Modern Higher Algebra* of Dr. Salmon, and the *Theorie der binären algebraischen Formen* of Clebsch; and in some degree to the *Théorie des Formes binaires* of the Chev. F. Faà De Bruno. We must record also our obligations in this department of the subject to Mr. Michael Roberts, from

* *Theory and Solution of Algebraical Equations*, London, 1835; *Analysis and Solution of Cubic and Biquadratic Equations*, London, 1842; and *Theory and Solution of Algebraical Equations of the Higher Orders*, London, 1843.

whose Papers in the *Quarterly Journal* and other periodicals, and from whose professorial lectures in the University of Dublin, very great assistance has been derived. Many of the examples also are taken from Papers set by him at the University Examinations.

In the Chapter on Complex Numbers and the Complex Variable we have followed closely the treatment of imaginary quantities given by M. Briot in his *Leçons d'Algèbre.*

In connexion with various parts of the subject several other works have been consulted, among which may be mentioned the treatises on Algebra by Serret, Meyer Hirsch, and Rubini, and papers in the mathematical journals by Boole, Cayley, Hermite, and Sylvester.

In the present edition we have introduced a short Chapter on the Covariants and Invariants of Combined Forms, and have added, in Notes at the end of the volume, an enumeration of the concomitants of the quintic and sextic. The section also of the last Chapter treating of Geometrical Transformations has been considerably enlarged. In the preparation of this edition we have received many valuable suggestions from Mr. Russell, Fellow of Trinity College, to whom we desire here to express our acknowledgments.

Trinity College,

May, 1892.

NOTE.—The first ten chapters of this work may be regarded as forming an elementary course. In reading these chapters for the first time, Students are recommended to omit Art. 53 of Chap. V., and to confine their attention in Chap. VI. to Arts. 55, 56, 57, 61, 62, 63, and 64.

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THEORY OF EQUATIONS.

INTRODUCTION.

1. **Definitions.**—Any mathematical expression involving a quantity is called a *function* of that quantity.

We shall be employed mainly with such algebraical functions as are *rational* and *integral.* By a *rational* function of a quantity is meant one which contains that quantity in a rational form only; that is, a form free from fractional indices or radical signs. By an *integral* function of a quantity is meant one in which the quantity enters in an integral form only; that is, never in the denominator of a fraction. The following expression, for example, in which n is a positive integer, is a *rational* and *integral algebraical function* of x:—

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + kx + l.$$

It is to be observed that this definition has reference to the quantity x only, of which the expression is regarded as a function. The several coefficients a, b, c, &c., may be irrational or fractional, and the function still remain rational and integral in x.

A function of x is represented for brevity by $F(x)$, $f(x)$, $\phi(x)$, or some such symbol.

The name *polynomial* is given to the algebraical function to express the fact that it is constituted of a number of terms

containing different powers of x connected by the signs plus or minus. For certain values of x regarded as variable one polynomial may become equal to another differently constituted. The algebraical expression of such a relation is called an *equation;* and any value of x which satisfies this equation is called a *root* of the equation. The determination of all possible roots constitutes the complete *solution of the equation.*

It is obvious that, by bringing all the terms to one side, we may arrange any equation according to descending powers of x in the following manner :—

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0.$$

The highest power of x in this equation being n, it is said to be an equation of the n^{th} *degree* in x. For such an equation we shall, in general, employ the form here written. The suffix attached to the letter a indicates the power of x which each coefficient accompanies, the sum of the exponent of x and the suffix of a being equal to n for each term. An equation is not altered if all its terms be divided by any quantity. We may thus, if we please, dividing by a_0, make the coefficient of x^n in the above equation equal to unity. It will often be found convenient to make this supposition; and in such cases the equation will be written in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

An equation is said to be *complete* when it contains terms involving x in all its powers from n to 0, and *incomplete* when some of the terms are absent; or, in other words, when some of the coefficients p_1, p_2, &c., are equal to zero. The term p_n, which does not contain x, is called the *absolute term.* An equation is *numerical* or *algebraical* according as its coefficients are numbers or algebraical symbols.

2. **Numerical and Algebraical Equations.**—In many researches in both mathematical and physical science the final mathematical problem presents itself in the form of an equation on whose solution that of the problem depends. It is natural,

therefore, that the attention of mathematicians should have been at an early stage in the history of the science directed towards inquiries of this nature. The science of the Theory of Equations, as it now stands, has grown out of the successive attempts of mathematicians to discover general methods for the solution of equations of any degree. When the coefficients of an equation are given numbers, the problem is to determine a numerical value, or perhaps several different numerical values, which will satisfy the equation. In this branch of the science very great progress has been made; and the best methods hitherto advanced for the discovery, either exactly or approximately, of the numerical values of the roots will be explained in their proper places in this work.

Equal progress has not been made in the general solution of equations whose coefficients are algebraical symbols. The student is aware that the root of an equation of the second degree, whose coefficients are such symbols, may be expressed in terms of these coefficients in a general formula; and that the numerical roots of any particular numerical equation may be obtained by substituting in this formula the particular numbers for the symbols. It was natural to inquire whether it was possible to discover any such formula for the solution of equations of higher degrees. Such results have been attained in the case of equations of the third and fourth degrees. It will be shown that in certain cases these formulas fail to supply the solution of a numerical equation by substitution of the numerical coefficients for the general symbols, and are, therefore, in this respect inferior to the corresponding algebraical solution of the quadratic.

Many attempts have been made to arrive at similar general formulas for equations of the fifth and higher degrees; but it may now be regarded as established by the researches of modern analysts that it is not possible by means of radical signs, and other signs of operation employed in common algebra, to express the root of an equation of the fifth or any higher degree in terms of the coefficients.

3. **Polynomials.**—From the preceding observations it is plain that one important object of the science of the Theory of Equations is the discovery of those values of the quantity x regarded as variable which give to the polynomial $f(x)$ the particular value zero. In attempting to discover such values of x we shall be led into many inquiries concerning the values assumed by the polynomial for other values of the variable. We shall, in fact, see in the next chapter that, corresponding to a continuous series of values of x varying from an infinitely great negative quantity $(-\infty)$ to an infinitely great positive quantity $(+\infty)$, $f(x)$ will assume also values continuously varying. The study of such variations is a very important part of the theory of polynomials. The general solution of numerical equations is, in fact, a tentative process; and by examining the values assumed by the polynomial for certain arbitrarily assumed values of the variable, we shall be led, if not to the root itself, at least to an indication of the neighbourhood in which it exists, and within which our further approximation must be carried on.

A polynomial is sometimes called a *quantic*. It is convenient to have distinct names for the quantics of various successive degrees. The terms *quadratic* (or *quadric*), *cubic*, *biquadratic* (or *quartic*), *quintic*, *sextic*, &c., are used to represent quantics of the 2nd, 3rd, 4th, 5th, 6th, &c., degrees; and the equations obtained by equating these quantics to zero are called *quadratic*, *cubic*, *biquadratic*, &c., *equations*, respectively.

CHAPTER I.

GENERAL PROPERTIES OF POLYNOMIALS.

4. In tracing the changes of value of a polynomial corresponding to changes in the variable, we shall first inquire what terms in the polynomial are most important when values very great or very small are assigned to x. This inquiry will form the subject of the present and succeeding Articles.

Writing the polynomial in the form

$$a_0 x^n \left\{1 + \frac{a_1}{a_0}\frac{1}{x} + \frac{a_2}{a_0}\frac{1}{x^2} + \ldots + \frac{a_{n-1}}{a_0}\frac{1}{x^{n-1}} + \frac{a_n}{a_0}\frac{1}{x^n}\right\},$$

it is plain that its value tends to become equal to $a_0 x^n$ as x tends towards ∞. The following theorem will determine a quantity such that the substitution of this, or of any greater quantity, for x will have the effect of making the term $a_0 x^n$ exceed the sum of all the others. In what follows we suppose a_0 to be positive; and in general in the treatment of polynomials and equations the highest term is supposed to be written with the positive sign.

Theorem.—*If in the polynomial*

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

the value $\dfrac{a_k}{a_0} + 1$, *or any greater value, be substituted for* x, *where* a_k *is that one of the coefficients* $a_1, a_2, \ldots a_n$ *whose numerical value is greatest, irrespective of sign, the term containing the highest power of* x *will exceed the sum of all the terms which follow.*

The inequality

$$a_0 x^n > a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

is satisfied by any value of x which makes

$$a_0 x^n > a_k (x^{n-1} + x^{n-2} + \ldots + x + 1),$$

where a_k is the greatest among the coefficients $a_1, a_2, \ldots a_{n-1}, a_n$ without regard to sign. Summing the geometric series within the brackets, we have

$$a_0 x^n > a_k \frac{x^n - 1}{x - 1}, \text{ or } x^n > \frac{a_k}{a_0 (x - 1)} (x^n - 1),$$

which is satisfied if $a_0 (x - 1)$ be $>$ or $= a_k$,

that is $$x > \text{or} = \frac{a_k}{a_0} + 1.$$

The theorem here proved is useful in supplying, when the coefficients of the polynomial are given numbers, a number such that when x receives values nearer to $+\infty$ the polynomial will preserve constantly a positive sign. If we change the sign of x, the first term will retain its sign if n be even, and will become negative if n be odd; so that the theorem also supplies a negative value of x, such that for any value nearer to $-\infty$ the polynomial will retain constantly a positive sign if n be even, and a negative sign if n be odd. The constitution of the polynomial is, in general, such that limits much nearer to zero than those here arrived at can be found beyond which the function preserves the same sign; for in the above proof we have taken the most unfavourable case, viz. that in which all the coefficients except the first are negative, and each equal to a_k; whereas in general the coefficients may be positive, negative, or zero. Several theorems, having for their object the discovery of such closer limits, will be given in a subsequent chapter.

5. We now proceed to inquire what is the most important term in a polynomial when the value of x is indefinitely diminished; and to determine a quantity such that the substitution of this, or of any smaller quantity, for x will have the effect of giving such term the preponderance.

Theorem.—*If in the polynomial*

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

the value $\dfrac{a_n}{a_n + a_k}$, or any smaller value, be substituted for x, where a_k is the greatest coefficient exclusive of a_n, the term a_n will be numerically greater than the sum of all the others.

To prove this, let $x = \dfrac{1}{y}$; then by the theorem of Art. 4, a_k being now the greatest among the coefficients $a_0, a_1, \ldots a_{n-1}$, without regard to sign, the value $\dfrac{a_k}{a_n} + 1$, or any greater value of y, will make

$$a_n y^n > a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \ldots + a_1 y + a_0,$$

that is,
$$a_n > a_{n-1} \frac{1}{y} + a_{n-2} \frac{1}{y^2} + \ldots a_0 \frac{1}{y^n};$$

hence the value $\dfrac{a_n}{a_n + a_k}$, or any less value of x, will make

$$a_n > a_{n-1} x + a_{n-2} x^2 + \ldots a_0 x^n.$$

This proposition is often stated in a different manner, as follows:—*Values so small may be assigned to x as to make the polynomial*

$$a_{n-1} x + a_{n-2} x^2 + \ldots + a_0 x^n$$

less than any assigned quantity.

This statement of the theorem follows at once from the above proof, since a_n may be taken to be the assigned quantity.

There is also another useful statement of the theorem, as follows:— *When the variable x receives a very small value, the sign of the polynomial*

$$a_{n-1} x + a_{n-2} x^2 + \ldots + a_0 x^n$$

is the same as the sign of its first term $a_{n-1} x$.

This appears by writing the expression in the form

$$x \{a_{n-1} + a_{n-2} x + \ldots + a_0 x^{n-1}\};$$

for when a value sufficiently small is given to x, the numerical value of the term a_{n-1} exceeds the sum of the other terms of the expression within the brackets, and the sign of that expression will consequently depend on the sign of a_{n-1}.

6. **Change of Form of a Polynomial corresponding to an increase or diminution of the Variable. Derived Functions.**—We shall now examine the form assumed by the polynomial when $x + h$ is substituted for x. If, in what follows, h be supposed essentially positive, the resulting form will correspond to an increase of the variable; and the form corresponding to a diminution of x will be obtained from this by changing the sign of h in the result.

When x is changed to $x + h$, $f(x)$ becomes $f(x + h)$, or

$$a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \dots + a_{n-1}(x+h) + a_n.$$

Let each term of this expression be expanded by the binomial theorem, and the result arranged according to ascending powers of h. We then have

$$
\begin{aligned}
& a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n \\
& + h\{na_0x^{n-1} + (n-1)\,a_1x^{n-2} + (n-2)\,a_2x^{n-3} + \dots + 2a_{n-2}x + a_{n-1}\} \\
& + \frac{h^2}{1.2}\{n\,(n-1)\,a_0x^{n-2} + (n-1)(n-2)\,a_1x^{n-3} + \dots + 2a_{n-2}\} \\
& + \quad . \quad . \quad . \quad . \quad . \quad . \\
& + \frac{h^n}{1.2.3\dots n}\{n\,.\,n-1\dots 2\,.\,1\}a_0.
\end{aligned}
$$

It will be observed that the part of this expression independent of h is $f(x)$ (a result obvious *à priori*), and that the successive coefficients of the different powers of h are functions of x of degrees diminishing by unity. It will be further observed that the coefficient of h may be derived from $f(x)$ in the following manner:—Let each term in $f(x)$ be multiplied by the exponent of x in that term, and let the exponent of x in the term be diminished by unity, the sign being retained; the sum of all the terms of $f(x)$ treated in this way will constitute a polynomial of dimensions one degree lower than those of $f(x)$. This polynomial is called the *first derived function* of $f(x)$. It is usual to represent this function by the notation $f'(x)$. The coefficient

of $\frac{h^2}{1 \cdot 2}$ may be derived from $f'(x)$ by a process the same as that employed in deriving $f'(x)$ from $f(x)$, or by the operation twice performed on $f(x)$. This coefficient is represented by $f''(x)$, and is called the *second derived function* of $f(x)$. In like manner the succeeding coefficients may all be derived by successive operations of this character; so that, employing the notation here indicated, we may write the result as follows:—

$$f(x+h) = f(x) + f'(x)\,h + \frac{f''(x)}{1 \cdot 2}\,h^2 + \frac{f'''(x)}{1 \cdot 2 \cdot 3}\,h^3 + \ldots + a_0 h^n.$$

It may be observed that, since the interchange of x and h does not alter $f(x+h)$, the expansion may also be written in the form

$$f(x+h) = f(h) + f'(h)\,x + \frac{f''(h)}{1 \cdot 2}\,x^2 + \frac{f'''(h)}{1 \cdot 2 \cdot 3}\,x^3 + \ldots + a_0 x^n.$$

We shall in general employ the notation here explained; but on certain occasions when it is necessary to deal with derived functions beyond the first two or three, it will be found more convenient to use suffixes instead of the accents here employed. The expansion will then be written as follows:—

$$f(x+h) = f(x) + f_1(x)h + f_2(x)\,\frac{h^2}{1 \cdot 2} + \ldots + f_r(x)\,\frac{h^r}{1 \cdot 2 \cdot 3 \ldots r} + \ldots$$

Example.

Find the result of substituting $x + h$ for x in the polynomial $4x^3 + 6x^2 - 7x + 4$.

Here

$$\begin{aligned} f(x) &= 4x^3 + 6x^2 - 7x + 4, \\ f'(x) &= 12x^2 + 12x - 7, \\ f''(x) &= 24x + 12, \\ f'''(x) &= 24; \end{aligned}$$

and the result is

$$4x^3 + 6x^2 - 7x + 4 + (12x^2 + 12x - 7)\,h + (24x + 12)\,\frac{h^2}{1 \cdot 2} + 24\,\frac{h^3}{1 \cdot 2 \cdot 3}.$$

The student may verify this result by direct substitution.

7. Continuity of a Rational Integral Function of x.— If in a rational and integral function $f(x)$ the value of x be

made to vary, by indefinitely small increments, from one quantity a to a greater quantity b, we proceed to prove that $f(x)$ at the same time varies also by indefinitely small increments; in other words, that $f(x)$ *varies continuously with* x.

Let x be increased from a to $a + h$. The corresponding increment of $f(x)$ is

$$f(a + h) - f(a);$$

and this is equal, by Art. 6, to

$$f'(a)h + f''(a)\frac{h^2}{1 . 2} + \ldots + a_0 h^n,$$

in which expression all the coefficients $f'(a)$, $f''(a)$, &c., are finite quantities. Now, by the theorem of Art. 5, this latter expression may, by taking h small enough, be made to assume a value less than any assigned quantity; so that the difference between $f(a + h)$ and $f(a)$ may be made as small as we please, and will ultimately vanish with h. The same is true during all stages of the variation of x from a to b; thus the continuity of the function $f(x)$ is established.

It is to be observed that it is not here proved that $f(x)$ *increases* continuously from $f(a)$ to $f(b)$. It may either increase or diminish, or at one time increase, and at another diminish; but the above proof shows that it cannot pass *per saltum* from one value to another; and that, consequently, amongst the values assumed by $f(x)$ while x increases continuously from a to b must be included all values between $f(a)$ and $f(b)$. The sign of $f'(a)$ will determine whether $f(x)$ is increasing or diminishing; for it appears by Art. 5 that when h is small enough the sign of the total increment will depend on that of $f'(a)h$. We thus observe that *when* $f'(a)$ is *positive* $f(x)$ *is increasing with* x*; and when* $f'(a)$ *is negative* $f(x)$ *is diminishing as* x *increases.*

8. **Form of the Quotient and Remainder when a Polynomial is divided by a Binomial.**—Let the quotient, when

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} \ldots + a_{n-1}x + a_n$$

is divided by $x - h$, be

$$b_0 x^{n-1} + b_1 x^{n-2} + \ldots + b_{n-2} x + b_{n-1}.$$

This we shall represent by Q, and the remainder by R. We have then the following equation :—

$$f(x) \equiv (x - h)\ Q + R.$$

The meaning of this equation is, that when Q is multiplied by $x - h$, and R added, the result must be *identical*, term for term, with $f(x)$. In order to distinguish equations of the kind here explained from equations which are not identities, it will often be found convenient to use the symbol here employed in place of the usual symbol of equality. The right-hand side of the identity is

$$b_0 x^n + \left.\begin{matrix} b_1 \\ -hb_0 \end{matrix}\right\} x^{n-1} + \left.\begin{matrix} b_2 \\ -hb_1 \end{matrix}\right\} x^{n-2} + \ldots + \left.\begin{matrix} b_{n-1} \\ -hb_{n-2} \end{matrix}\right\} x + \begin{matrix} R \\ -hb_{n-1}. \end{matrix}$$

Equating the coefficients of x on both sides, we get the following series of equations to determine $b_0, b_1, b_2, \ldots b_{n-1}, R$:—

$$\begin{aligned}
b_0 &= a_0, \\
b_1 &= b_0 h + a_1, \\
b_2 &= b_1 h + a_2, \\
b_3 &= b_2 h + a_3, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
b_{n-1} &= b_{n-2} h + a_{n-1}, \\
R &= b_{n-1} h + a_n.
\end{aligned}$$

These equations supply a ready method of calculating in succession the coefficients b_0, b_1, &c. of the quotient, and the remainder R. For this purpose we write the series of operations in the following manner :—

$$\begin{array}{ccccccc}
a_0, & a_1, & a_2, & a_3, & \ldots\ldots & a_{n-1}, & a_n, \\
 & b_0 h, & b_1 h, & b_2 h, & \ldots\ldots & b_{n-2} h, & b_{n-1} h, \\
\hline
 & b_1, & b_2, & b_3, & \ldots\ldots & b_{n-1}, & R.
\end{array}$$

In the first line are written down the successive coefficients

of $f(x)$. The first term in the second line is obtained by multiplying a_0 (or b_0, which is equal to it) by h. The product b_0h is placed under a_1, and then added to it in order to obtain the term b_1 in the third line. This term, when obtained, is multiplied in its turn by h, and placed under a_2. The product is added to a_2 to obtain the second figure b_2 in the third line. The repetition of this process furnishes in succession all the coefficients of the quotient, the last figure thus obtained being the remainder. A few examples will make this plain.

EXAMPLES.

1. Find the quotient and remainder when $3x^4 - 5x^3 + 10x^2 + 11x - 61$ is divided by $x - 3$.

The calculation is arranged as follows:—

$$\begin{array}{rrrrr} 3 & -5 & 10 & 11 & -61. \\ & 9 & 12 & 66 & 231. \\ \hline & 4 & 22 & 77 & 170. \end{array}$$

Thus the quotient is $3x^3 + 4x^2 + 22x + 77$, and the remainder 170.

2. Find the quotient and remainder when $x^3 + 5x^2 + 3x + 2$ is divided by $x - 1$.

Ans. $Q = x^2 + 6x + 9$, $R = 11$.

3. Find Q and R when $x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x - 5$.

N.B.—When any term in a polynomial is absent, care must be taken to supply the place of its coefficient by zero in writing down the coefficients of $f(x)$. In this example, therefore, the series in the first line will be

$$1 \quad -4 \quad 7 \quad 0 \quad -11 \quad -13.$$

Ans. $Q = x^4 + x^3 + 12x^2 + 60x + 289$; $R = 1432$.

4. Find Q and R when $x^9 + 3x^7 - 15x^2 + 2$ is divided by $x - 2$.

Ans. $Q = x^8 + 2x^7 + 7x^6 + 14x^5 + 28x^4 + 56x^3 + 112x^2 + 209x + 418$; $R = 838$.

5. Find Q and R when $x^5 + x^2 - 10x + 113$ is divided by $x + 4$.

Ans. $Q = x^4 - 4x^3 + 16x^2 - 63x + 242$; $R = -855$.

9. **Tabulation of Functions.**—The operation explained in the preceding Article affords a convenient practical method of calculating the numerical value of a polynomial whose coefficients are given numbers when any number is substituted for x. For, the equation

$$f(x) = (x - h)\,Q + R,$$

since its two members are identically equal, must be satisfied

when any quantity whatever is substituted for x. Let $x = h$, then $f(h) = R$, $x - h$ being $= 0$, and Q remaining finite. Hence the result of substituting h for x in $f(x)$ is the remainder when $f(x)$ is divided by $x - h$, and can be calculated rapidly by the process of the last Article.

For example, the result of substituting 3 for x in the polynomial of Ex. 1, Art. 8, viz.,

$$3x^4 - 5x^3 + 10x^2 + 11x - 61,$$

is 170, this being the remainder after division by $x - 3$. The student can verify this by actual substitution.

Again, the result of substituting -4 for x in

$$x^5 + x^2 - 10x + 113$$

is -855, as appears from Ex. 5, Art. 8. We saw in Art. 7 that as x receives a continuous series of values increasing from $-\infty$ to $+\infty$, $f(x)$ will pass through a corresponding continuous series. If we substitute in succession for x, in a polynomial whose coefficients are given numbers, a series of numbers such as

$$\ldots -5,\ -4,\ -3,\ -2,\ -1,\quad 0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\ \ldots,$$

and calculate the corresponding values of $f(x)$, the process may be called the *tabulation of the function.*

Examples.

1. Tabulate the trinomial $2x^2 + x - 6$, for the following values of x:—

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

Values of x,	-4	-3	-2	-1	0	1	2	3	4
,, ,, $f(x)$,	22	9	0	-5	-6	-3	4	15	30

2. Tabulate the polynomial $10x^3 - 17x^2 + x + 6$ for the same values of x.

Values of x,	-4	-3	-2	-1	0	1	2	3	4
,, ,, $f(x)$,	-910	-420	-144	-22	6	0	20	126	378

10. **Graphic Representation of a Polynomial.**—In investigating the changes of a function $f(x)$ consequent on any

series of changes in the variable which it contains, it is plain that great advantage will be derived from any mode of representation which renders possible a rapid comparison with one another of the different values which the function may assume. In the case where the function in question is a polynomial with numerical coefficients, to any assumed value of x will correspond one definite value of $f(x)$. We proceed to explain a mode of graphic representation by which it is possible to exhibit to the eye the several values of $f(x)$ corresponding to the different values of x.

Let two right lines OX, OY (fig. 1) cut one another at right angles, and be produced indefinitely in both directions. These lines are called the *axis of x* and *axis of y*, respectively. Lines, such as OA, measured on the axis of x at the right-hand side of O, are regarded as positive; and those, such as OA', measured at the left-hand side, as negative. Lines parallel to OY which are above XX', such as AP or $B'Q'$, are positive; and those below it, such as AT or $A'P'$, are negative. These conventions are already familiar to the student acquainted with Trigonometry.

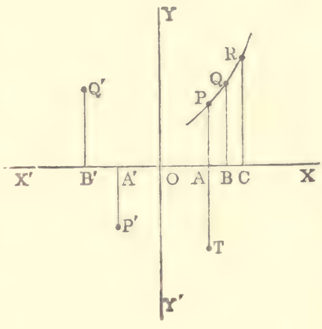

Fig. 1.

Any arbitrary length may now be taken on OX as unity, and any number positive or negative will be represented by a line measured on XX'; the series of numbers increasing from 0 to $+\infty$ in the direction OX, and diminishing from 0 to $-\infty$ in the direction OX'. Let any number m be represented by OA; calculate $f(m)$; from A draw AP parallel to OY to represent $f(m)$ in magnitude on the same scale as that on which OA represents m, and to represent by its position above or below the line OX the sign of $f(m)$. Corresponding to the different values of m represented by OA, OB, OC, &c., we shall have a series of points P, Q, R, &c., which, when we suppose the series of values of

m indefinitely increased so as to include all numbers between $-\infty$ and $+\infty$, will trace out a continuous curved line. This curve will, by the distances of its several points from the line OX, exhibit to the eye the several values of the function $f(x)$.

The process here explained is also called *tracing the function* $f(x)$. The student acquainted with analytic geometry will observe that it is equivalent to tracing the plane curve whose equation is $y = f(x)$.

In the practical application of this method it is well to begin by laying down the points on the curve corresponding to certain small integral values of x, positive and negative. It will then in general be possible to draw through these points a curve which will exhibit the progress of the function, and give a general idea of its character. The accuracy of the representation will of course increase with the number of points determined between any two given values of the variable. When any portion of the curve between two proposed limits has to be examined with care, it will often be necessary to substitute values of the variable separated by smaller intervals than unity. The following examples will illustrate these principles.

EXAMPLES.

1. Trace the trinomial $2x^2 + x - 6$.

The unit of length taken is one-sixth of the line OD in fig. 2.

In Ex. 1, Art. 9, the values of $f(x)$ are given corresponding to the integral values of x from -4 to $+4$, inclusive.

By means of these values we obtain the positions of nine points on the curve; seven of which, A, B, C, D, E, F, G, are here represented, the other two corresponding to values of $f(x)$ which lie out of the limits of the figure.

The student will find it a useful exercise to trace the curve more minutely between the points C and E in the figure, viz. by calculating the values of $f(x)$ corresponding to all values of x between -1 and 1 separated by small intervals, say of one-tenth, as is done in the following example.

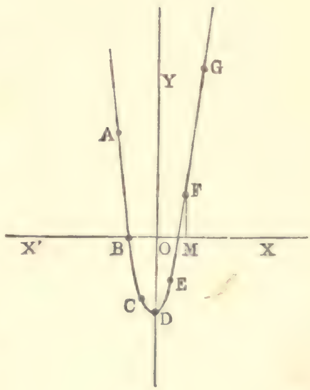

Fig. 2.

2. Trace the polynomial

$$10x^3 - 17x^2 + x + 6.$$

This is already tabulated in Art. 9 for values of x between -4 and 4.

It may be observed, as an exercise on Art. 4, that this function retains positive values for all positive values of x greater than $2{\cdot}7$, and negative values for all values of x nearer to $-\infty$ than $-2{\cdot}7$. The curve will, then, if it cuts the axis of x at all, cut it at a point (or points) corresponding to some value (or values) of x between $-2{\cdot}7$ and $+2{\cdot}7$; so that if our object is to determine, or approximate to, the positions of the roots of the equation $f(x) = 0$, the tabulation may be confined to the interval between $-2{\cdot}7$ and $2{\cdot}7$.

This is a case in which the substitution of integral values only of x gives very little help towards the tracing of the curve, and where, consequently, smaller intervals have to be examined. We give the tabulation of the function for intervals of one-tenth between the integers $-1, 0$; $0, 1$; $1, 2$. From these values the positions of the corresponding points on the curve may be approximately ascertained, and the curve traced as in fig. 3.

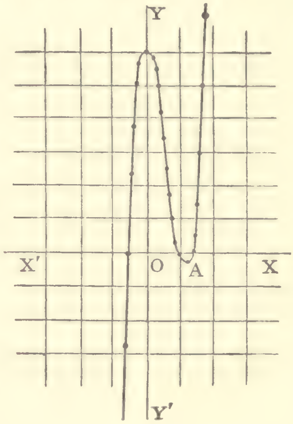

Fig. 3.

Values of x	-1	$-{\cdot}9$	$-{\cdot}8$	$-{\cdot}7$	$-{\cdot}6$	$-{\cdot}5$	$-{\cdot}4$	$-{\cdot}3$	$-{\cdot}2$	$-{\cdot}1$
,, ,, $f(x)$	-22	$-15{\cdot}96$	$-10{\cdot}8$	$-6{\cdot}46$	$-2{\cdot}88$	0	$2{\cdot}24$	$3{\cdot}9$	$5{\cdot}04$	$5{\cdot}72$

Values of x	0	$\cdot 1$	$\cdot 2$	$\cdot 3$	$\cdot 4$	$\cdot 5$	$\cdot 6$	$\cdot 7$	$\cdot 8$	$\cdot 9$
,, ,, $f(x)$	6	$5{\cdot}94$	$5{\cdot}6$	$5{\cdot}04$	$4{\cdot}32$	$3{\cdot}5$	$2{\cdot}64$	$1{\cdot}8$	$1{\cdot}04$	$\cdot 42$

Values of x	1	$1{\cdot}1$	$1{\cdot}2$	$1{\cdot}3$	$1{\cdot}4$	$1{\cdot}5$	$1{\cdot}6$	$1{\cdot}7$	$1{\cdot}8$	$1{\cdot}9$	2
,, ,, $f(x)$	0	$-{\cdot}16$	0	$\cdot 54$	$1{\cdot}52$	3	$5{\cdot}04$	$7{\cdot}7$	$11{\cdot}04$	$15{\cdot}12$	20

The curve traced in Ex. 1 cuts the axis of x in two points (a number equal to the degree of the polynomial): in other words, there are two values of x for which the value of the given polynomial is zero; these are the roots of the equation $2x^2 + x - 6 = 0$, viz. -2, and $1{\cdot}5$. Similarly, the curve traced in Ex. 2 cuts the axis in three points, viz. the points corresponding to the roots of the cubic equation $10x^3 - 17x^2 + x + 6 = 0$. The curve

representing a given polynomial may not cut the axis of x at all, or may cut it in a number of points less than the degree of the polynomial. Such cases correspond to the imaginary roots of equations, as will appear more fully in the next chapter. For example, the curve which represents the polynomial $2x^2 + x + 2$ will, when traced, lie entirely above the axis of x; in fact, since this function differs from the function of Ex. 1 only by the addition of the constant quantity 8, each value of $f(x)$ is obtained by adding 8 to the previously calculated value, and the entire curve can be obtained by simply supposing the previously traced curve to be moved up parallel to the axis of y through a distance equal to 8 of the units. It is evident, by the solution of the equation $2x^2 + x + 2 = 0$, that the two values of x which render the polynomial zero are in this case imaginary. Whenever the number of points in which the curve cuts the axis of x falls short of the degree of the polynomial, it is customary to speak of the curve as *cutting the line in imaginary points*.

11. **Maximum and Minimum Values of Polynomials.** —It is apparent from the considerations established in the preceding Articles, that as the variable x changes from $-\infty$ to $+\infty$, the function $f(x)$ may undergo many variations. It may go on for a certain period increasing, and then, ceasing to increase, may commence to diminish; it may then cease to diminish and commence again to increase; after which another period of diminution may arrive, or the function may (as in the last example of the preceding Art.) go on then continually increasing. At a stage where the function ceases to increase and commences to diminish, it is said to have attained a *maximum* value; and when it ceases to diminish and commences to increase, it is said to have attained a *minimum* value. A polynomial may have several such values; the number depending in general on the degree of the function. Nothing exhibits so well as a graphic representation the occurrence of such a maximum or minimum value; as well as the various fluctuations of which the values of a polynomial are susceptible.

A knowledge of the maximum and minimum values of a function, giving the positions of the points where the curve bends with reference to the axis, is often of great assistance in tracing the curve corresponding to a given polynomial. It will be shown in a subsequent chapter that the determination of these points depends on the solution of an equation one degree lower than that of the given function.

It is easy to show that maxima and minima occur alternately; for, as the variable increases from a value corresponding to one maximum to the value corresponding to a second, the function begins by diminishing and ends by increasing, and therefore attains a minimum at some intermediate stage. In like manner it appears that between two minima one maximum must exist.

CHAPTER II.

GENERAL PROPERTIES OF EQUATIONS.

12. The process of tracing the function $f(x)$ explained in Art. 10 may be employed for the purpose of ascertaining approximately the real roots of a given numerical equation; for when the corresponding curve is accurately traced, the real roots of the equation $f(x) = 0$ can be obtained approximately by measuring the distances from the origin of its points of intersection with the axis. With a view to the more accurate numerical solution of this problem, as well as the general discussion of equations both numerical and algebraical, we proceed to establish in the present chapter the most important general properties of equations having reference to the existence and number of the roots, and the distinction between real and imaginary roots.

By the aid of the following theorem the existence of a real root in an equation may often be established:—

Theorem.—*If two real quantities a and b be substituted for the unknown quantity x in any polynomial $f(x)$, and if they furnish results having different signs, one plus and the other minus; then the equation $f(x) = 0$ must have at least one real root intermediate in value between a and b.*

This theorem is an immediate consequence of the property of the continuity of the function $f(x)$ established in Art. 7; for since $f(x)$ changes continuously from $f(a)$ to $f(b)$, and therefore passes through all the intermediate values, while x changes from a to b; and since one of these quantities, $f(a)$ or $f(b)$, is positive, and the other negative, it follows that for some value of x intermediate between a and b, $f(x)$ must attain the value zero which is intermediate between $f(a)$ and $f(b)$.

The student will assist his conception of this theorem by reference to the graphic method of representation. What is here proved, and what will appear obvious from the figure, is, that if there exist two points of the curved line representing the polynomial on opposite sides of the axis OX, then the curve joining these points must cut that axis at least once. It will also be evident from the figure that several values may exist between a and b for which $f(x) = 0$, *i.e.* for which the curve cuts the axis. For example, in fig. 3, Art. 10, $x = -2$ gives a negative value (-144), and $x = 2$ gives a positive value (20), and between these points of the curve there exist *three* points of section of the axis of x.

Corollary.—*If there exist no real quantity which, substituted for x, makes $f(x) = 0$, then $f(x)$ must be positive for every real value of x.*

For it is evident (Art. 4) that $x = \infty$ makes $f(x)$ positive; and no value of x, therefore, can make it negative; for if there were any such value, the equation would by the theorem of this Article have a real root, which is contrary to our present hypothesis. With reference to the graphic mode of representation this theorem may be expressed by saying that when the equation $f(x) = 0$ has no real root, the curve representing the polynomial $f(x)$ must lie entirely above the axis of x.

13. **Theorem.**—*Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

This is an immediate consequence of the theorem in the last Article. Substitute in succession $-\infty$, 0, ∞ for x in the polynomial $f(x)$. The results are, n being odd (see Art. 4),

for $x = -\infty$, $f(x)$ is negative;

,, $x = 0$, sign of $f(x)$ is the same as that of a_n;

,, $x = +\infty$, $f(x)$ is positive.

If a_n is positive, the equation must have a real root between $-\infty$ and 0, *i.e.* a real negative root; and if a_n is negative, the

equation must have a real root between 0 and ∞, *i.e.* a real positive root. The theorem is therefore proved.

14. **Theorem.**—*Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

The results of substituting $-\infty$, 0, ∞ are in this case

$$\begin{array}{cc} -\infty, & +, \\ 0, & -, \\ +\infty, & +; \end{array}$$

hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$; *i.e.* there exist at least one real negative, and one real positive root.

We have contented ourselves in both this and the preceding Articles with proving the *existence* of roots, and for this purpose it is sufficient to substitute very large positive or negative values, as we have done, for x. It is of course possible to narrow the limits within which the roots lie by the aid of the theorem of Art. 4, and still more by the aid of the theorems respecting the limits of the roots to be given in a subsequent chapter.

15. **Existence of a Root in the General Equation. Imaginary Roots.**—We have now proved the existence of a real root in the case of every equation except one of an even degree whose last term is positive. Such an equation may have no real root at all. It is necessary then to examine whether, in the absence of real values, there may not be values involving the imaginary expression $\sqrt{-1}$, which, when substituted for x, reduce the polynomial to zero; or whether there may not be in certain cases both real and imaginary values of the variable which satisfy the equation. We take a simple

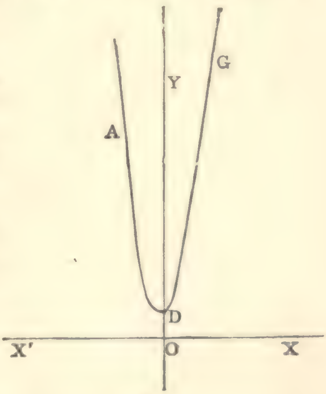

Fig. 4.

example to illustrate the occurrence of such imaginary roots. As already remarked (Art. 10), the curve corresponding to the polynomial

$$f(x) = 2x^2 + x + 2$$

lies entirely above the axis of x, as in fig. 4. The equation $f(x) = 0$ has no real roots; but it has the two imaginary roots

$$-\frac{1}{4} + \frac{\sqrt{15}}{4}\sqrt{-1}, \quad -\frac{1}{4} - \frac{\sqrt{15}}{4}\sqrt{-1},$$

as is evident by the solution of the quadratic. We observe, therefore, that in the absence of any real values there are in this case two imaginary expressions which reduce the polynomial to zero.

The corresponding general proposition is, that *Every rational integral equation has a root of the form*

$$\alpha + \beta\sqrt{-1},$$

α and β being real finite quantities. This statement includes both real and imaginary roots, the former corresponding to the value $\beta = 0$. When α and β are numbers, such an expression is called a *complex number;* and what is asserted is that every numerical equation has a numerical root either real or complex.

As the proof of this proposition involves principles which could not conveniently have been introduced hitherto, and which will present themselves more naturally for discussion in subsequent parts of the work, we defer the demonstration until these principles have been established. For the present, therefore, we assume the proposition, and proceed to derive certain consequences from it.

16. **Theorem**.—*Every equation of n dimensions has n roots, and no more.*

We first observe that if any quantity h is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $x - h$ without a remainder. This is evident from Art. 9; for if $f(h) = 0$, *i.e.* if h is a root of $f(x) = 0$, R must be $= 0$.

Let, now, the given equation be

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0.$$

This equation must have a root, real or imaginary (Art. 15), which we shall denote by the symbol a_1. Let the quotient, when $f(x)$ is divided by $x - a_1$, be $\phi_1(x)$; we have then the identical equation

$$f(x) \equiv (x - a_1)\,\phi_1(x).$$

Again, the equation $\phi_1(x) = 0$, which is of $n - 1$ dimensions, must have a root, which we represent by a_2. Let the quotient obtained by dividing $\phi_1(x)$ by $x - a_2$ be $\phi_2(x)$. Hence

$$\phi_1(x) \equiv (x - a_2)\,\phi_2(x),$$

$$\text{and } \therefore f(x) \equiv (x - a_1)(x - a_2)\,\phi_2(x),$$

where $\phi_2(x)$ is of $n - 2$ dimensions.

Proceeding in this manner, we prove that $f(x)$ consists of the product of n factors, each containing x in the first degree, and a numerical factor $\phi_n(x)$. Comparing the coefficients of x^n, it is plain that $\phi_n(x) = 1$. Thus we prove the identical equation

$$f(x) \equiv (x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_{n-1})(x - a_n).$$

It is evident that the substitution of any one of the quantities $a_1, a_2, \ldots a_n$ for x in the right-hand member of this equation will reduce that member to zero, and will therefore reduce $f(x)$ to zero; that is to say, the equation $f(x) = 0$ has for roots the n quantities $a_1, a_2, a_3 \ldots a_{n-1}, a_n$. And it can have no other roots; for if any quantity other than one of the quantities $a_1, a_2, \ldots a_n$ be substituted in the right-hand member of the above equation, the factors will be all different from zero, and therefore the product cannot vanish.

Corollary.—*Two polynomials each of the n^{th} degree in x cannot be equal to one another for more than n values of x without being completely identical.*

For if their difference be equated to zero, we obtain an equation of the n^{th} degree, which can be satisfied by n values only of x, unless each coefficient be separately equal to zero.

The theorem of this Article, although of no assistance in the solution of the equation $f(x) = 0$, enables us to solve completely the converse problem, *i.e.* to find the equation whose roots are any n given quantities. The required equation is obtained by multiplying together the n simple factors formed by subtracting from x each of the given roots. By the aid of the present theorem also, when any (one or more) of the roots of a given equation are known, the equation containing the remaining roots may be obtained. For this purpose it is only necessary to divide the given equation by the product of the given binomial factors. The quotient will be the required polynomial composed of the remaining factors.

Examples.

1. Find the equation whose roots are

$$-3, \quad -1, \quad 4, \quad 5.$$

Ans. $x^4 - 5x^3 - 13x^2 + 53x + 60 = 0.$

2. The equation

$$x^4 - 6x^3 + 8x^2 - 17x + 10 = 0$$

has a root 5; find the equation containing the remaining roots.

Use the method of division of Art. 8.

Ans. $x^3 - x^2 + 3x - 2 = 0.$

3. Solve the equation

$$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0,$$

two roots being 1 and 7.

Ans. The other two roots are 3, 5.

4. Form the equation whose roots are

$$-\frac{3}{2}, \quad 3, \quad \frac{1}{7}.$$

Ans. $14x^3 - 23x^2 - 60x + 9 = 0.$

5. Solve the cubic equation

$$x^3 - 1 = 0.$$

Here it is evident that $x = 1$ satisfies the equation. Divide by $x - 1$, and solve the resulting quadratic. The two roots are found to be

$$-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

6. Form an equation with rational coefficients which shall have for a root the irrational expression

$$\sqrt{p} + \sqrt{q}.$$

This expression has four different values according to the different combinations of the radical signs, viz.

$$\sqrt{p} + \sqrt{q}, \quad -\sqrt{p} - \sqrt{q}, \qquad \sqrt{p} - \sqrt{q}, \quad -\sqrt{p} + \sqrt{q}.$$

The required equation is, therefore,

$$(x - \sqrt{p} - \sqrt{q})(x + \sqrt{p} + \sqrt{q})(x - \sqrt{p} + \sqrt{q})(x + \sqrt{p} - \sqrt{q}) = 0,$$

or

$$(x^2 - p - q - 2\sqrt{pq})(x^2 - p - q + 2\sqrt{pq}) = 0,$$

or, finally,

$$x^4 - 2(p + q)x^2 + (p - q)^2 = 0.$$

17. **Equal Roots.**—It must be observed that the n factors of which a polynomial $f(x)$ consists need not be all different from one another. The factor $x - a$, for example, may occur in the second, or any higher power not superior to n. In this case the equation $f(x) = 0$ is still said to have n roots, two or more being now equal to one another; and the root a is called a multiple root of the equation—double, triple, &c., according to the number of times the factor is repeated.

A reference to the graphic construction in Art. 10 (fig. 3) will help to explain the occurrence of multiple roots. We see by an inspection of the figure that the two positive roots of the equation $10x^3 - 17x^2 + x + 6 = 0$ are nearly equal, and we may conceive that a slight addition to the absolute term of this polynomial, which is, as already explained, equivalent to a small parallel movement upwards of the whole curve, would have the effect of rendering equal the roots of the equation thus altered. In that case the line OX would no longer cut the curve in two distinct points, but would *touch* it. Now, when a line touches a curve it is properly said to meet the curve, not once, but in *two coincident points*. The student acquainted with the theory of plane curves will have no difficulty in illustrating in a similar manner the occurrence of a triple or higher multiple root.

Equal roots form the connecting link between real and imaginary roots. We have just seen that a small change in the form of a polynomial may convert it from one having real roots into another in which two of the real roots become equal. A further small change may convert it into a form in which the

two roots become imaginary. Let us suppose that the above polynomial is further altered by another small addition to the absolute term. We shall then have a graphic representation in which the axis OX cuts the curve in only one real point, viz. that corresponding to the negative root, the two points of section corresponding to the two positive roots having now disappeared.

Consider, for example, the polynomial $10x^3 - 17x^2 + x + 28$, which is obtained from that of Ex. 2, Art. 10, by the addition of 22. The student can easily construct the figure; the point corresponding to A in fig. 3 will now lie much above the axis of x. Divide by $x + 1$, and obtain the trinomial $10x^2 - 27x + 28$ which contains the remaining two roots. They are easily found to be

$$\frac{27}{20} + \frac{\sqrt{391}}{20}\sqrt{-1}, \quad \frac{27}{20} - \frac{\sqrt{391}}{20}\sqrt{-1}.$$

We observe in this case, as well as in the example of Art. 15, that when a change of form of the polynomial causes one real root to disappear, a second also disappears at the same time, and the two are replaced by a pair of imaginary roots. The reason of this will be apparent from the proposition of the following Article.

18. **Imaginary Roots enter Equations in Pairs.**— The proposition to be now proved may be stated as follows:—

If an equation $f(x) = 0$, whose coefficients are all real quantities, have for a root the imaginary expression $\alpha + \beta\sqrt{-1}$, it must also have for a root the conjugate imaginary expression $\alpha - \beta\sqrt{-1}$.

We have the following identity:—

$$(x - \alpha - \beta\sqrt{-1})(x - \alpha + \beta\sqrt{-1}) \equiv (x - \alpha)^2 + \beta^2.$$

Let the polynomial $f(x)$ be divided by the second member of this identity, and if possible let there be a remainder $Rx + R'$. We have then the identical equation

$$f(x) \equiv \{(x - \alpha)^2 + \beta^2\}\,Q + Rx + R',$$

where Q is the quotient, of $n - 2$ dimensions in x. Substitute in

this identity $a + \beta\sqrt{-1}$ for x. This, by hypothesis, causes $f(x)$ to vanish. It also causes $(x - a)^2 + \beta^2$ to vanish. Hence

$$R(a + \beta\sqrt{-1}) + R' = 0,$$

from which we obtain the two equations

$$Ra + R' = 0, \qquad R\beta = 0,$$

since the real and imaginary parts cannot destroy one another; hence

$$R = 0, \qquad R' = 0.$$

Thus the remainder $Rx + R'$ vanishes; and, therefore, $f(x)$ is divisible without remainder by the product of the two factors

$$x - a - \beta\sqrt{-1}, \qquad x - a + \beta\sqrt{-1}.$$

The equation has, consequently, the root $a - \beta\sqrt{-1}$ as well as the root $a + \beta\sqrt{-1}$.

Thus the total number of imaginary roots in an equation with real coefficients is always even; and every polynomial may be regarded as composed of real factors, each pair of imaginary roots producing a real quadratic factor, and each real root producing a real simple factor. The actual resolution of the polynomial into these factors constitutes the complete solution of the equation.

We observed in Art. 17 that equal roots may be considered as the connecting link between real and imaginary roots. This statement may now be regarded from another point of view. Suppose a polynomial has the quadratic factor $(x - a)^2 + k$, and let its form be altered by means of slight alterations in the value of k. When k is negative, the quadratic factor gives a pair of *real* roots; when $k = 0$, this factor has two *equal* roots, a; when k is positive, the factor has two *imaginary* roots.

A proof exactly similar to that above given shows that *surd roots, of the form $a \pm \sqrt{\gamma}$, enter equations whose coefficients are rational in pairs.*

EXAMPLES.

1. Form a rational cubic equation which shall have for roots

$$1, \quad 3 + 2\sqrt{-1}.$$

Ans. $x^3 - 7x^2 + 19x - 13 = 0$.

2. Form a rational equation which shall have for two of its roots

$$1 + 5\sqrt{-1}, \quad 5 - \sqrt{-1}.$$

Ans. $x^4 - 12x^3 + 72x^2 - 312x + 676 = 0$.

3. Solve the equation

$$x^4 + 2x^3 - 5x^2 + 6x + 2 = 0,$$

which has a root

$$-2 + \sqrt{3}.$$

Ans. The roots are $-2 \pm \sqrt{3}, \quad 1 \pm \sqrt{-1}$.

4. Solve the equation

$$3x^3 - 4x^2 + x + 88 = 0,$$

one root being

$$2 + \sqrt{-7}.$$

Ans. The roots are $2 \pm \sqrt{-7}. \quad -\frac{8}{3}$.

19. **Descartes' Rule of Signs—Positive Roots.**—This rule, which enables us, by the mere inspection of a given equation, to assign a superior limit to the number of its positive roots, may be enunciated as follows:—*No equation can have more positive roots than it has changes of sign from + to –, and from – to +, in the terms of its first member.*

We shall content ourselves for the present with the proof which is usually given, and which is rather a verification than a general demonstration of this celebrated theorem of Descartes. It will be subsequently shown that the rule just enunciated, and other similar rules which were discovered by early investigators relative to the number of the positive, negative, and imaginary roots of equations, are immediate deductions from the more general theorems of Budan and Fourier.

Let the signs of a polynomial taken at random succeed each other in the following order:—

$$+ + - + - - - + + - + -.$$

In this there are in all seven changes of sign, including changes from + to –, and from – to +. It is proposed to show

that if this polynomial be multiplied by a binomial whose signs, corresponding to a positive root, are $+\ -$, the resulting polynomial will have at least one more change of sign than the original.

We write down only the signs which occur in the operation as follows:—

$$\begin{array}{l} + + - + - - - + + - + - \\ \quad - - + - + + + - - + - + \\ \hline + \pm - + - \mp \mp + \pm - + - + \end{array}$$

Here, in the third line, the ambiguous sign $\pm$ is placed wherever there are two terms with different signs to be added. We observe in this case, and it will readily appear also for every other arrangement, that the effect of the process is to introduce the ambiguous sign wherever the sign $+$ follows $+$, or $-$ follows $-$, in the original polynomial. The number of variations of sign is never diminished. There is, moreover, always one variation added at the end. This is obvious in the above instance, where the original polynomial terminates with a variation; if it terminate with a continuation of sign, it will equally appear that the corresponding ambiguity in the resulting polynomial must furnish one additional variation either with the preceding or with the superadded sign. Thus, in even the most unfavourable case—that, namely, in which the continuations of sign in the original remain continuations in the resulting polynomial, there is one variation added; and we may conclude in general that the effect of the multiplication of a polynomial by a binomial factor $x - a$ is to introduce at least one additional change of sign.

Suppose now a polynomial formed of the product of the factors corresponding to the negative and imaginary roots of an equation; the effect of multiplying this by each of the factors $x - \alpha$, $x - \beta$, $x - \gamma$, &c., corresponding to the positive roots α, β, γ, &c., is to introduce at least one change of sign for each; so that when the complete product is formed containing

all the roots, we conclude that the resulting polynomial has at least as many changes of sign as it has positive roots. This is Descartes' proposition.

20. **Descartes' Rule of Signs—Negative Roots.**—In order to give the most advantageous statement to Descartes' rule in the case of negative roots, we first prove that if $-x$ be substituted for x in the equation $f(x) = 0$, the resulting equation will have the same roots as the original except that their signs will be changed. This follows from the identical equation of Art. 16

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \ldots . (x - a_n),$$

from which we derive

$$f(-x) \equiv (-1)^n (x + a_1)(x + a_2)(x + a_3) \ldots . (x + a_n).$$

From this it is evident that the roots of $f(-x) = 0$ are

$$-a_1, \quad -a_2, \quad -a_3, \ldots . -a_n.$$

Hence the negative roots of $f(x)$ are positive roots of $f(-x)$, and we may enunciate Descartes' rule for negative roots as follows:—*No equation can have a greater number of negative roots than there are changes of sign in the terms of the polynomial $f(-x)$.*

21. **Use of Descartes' Rule in proving the existence of Imaginary Roots.**—It is often possible to detect the existence of imaginary roots in equations by the application of Descartes' rule; for if it should happen that the sum of the greatest possible number of positive roots, added to the greatest possible number of negative roots, is less than the degree of the equation, we are sure of the existence of imaginary roots. Take, for example, the equation

$$x^8 + 10x^3 + x - 4 = 0.$$

This equation, having only one variation, cannot have more than one positive root. Now, changing x into $-x$, we get

$$x^8 - 10x^3 - x - 4 = 0;$$

and since this has only one variation, the original equation cannot have more than one negative root. Hence, in the proposed

equation there cannot exist more than two real roots. It has, therefore, at least six imaginary roots. This application of Descartes' rule is available only in the case of incomplete equations; for it is easily seen that the sum of the number of variations in $f(x)$ and $f(-x)$ is exactly equal to the degree of the equation when it is complete.

22. **Theorem.**—*If two numbers a and b, substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x) = 0$ lies between them; and if they give results with the same sign, either no real root or an even number of real roots lies between them.*

This proposition, of which the theorem in Art. 12 is a particular case, contains in the most general form the conclusions which can be drawn as to the roots of an equation from the signs furnished by its first member when two given numbers are substituted for x. We proceed to prove the first part of the proposition; the second part is proved in a precisely similar manner.

Let the following m roots $\alpha_1, \alpha_2, \ldots \alpha_m$, and no others, of the equation $f(x) = 0$ lie between the quantities a and b, of which, as usual, we take a to be the lesser.

Let $\phi(x)$ be the quotient when $f(x)$ is divided by the product of the m factors $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)$. We have, then, the identical equation

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)\,\phi(x).$$

Putting in this successively $x = a$, $x = b$, we obtain

$$f(a) = (a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m)\,\phi(a),$$
$$f(b) = (b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m)\,\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ have the same sign; for if they had different signs there would be, by Art. 12, one root at least of the equation $\phi(x) = 0$ between them. By hypothesis, $f(a)$ and $f(b)$ have different signs; hence the signs of the products

$$(a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m),$$
$$(b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m),$$

are different; but the sign of the second is positive, since all its factors are positive; hence the sign of the first is negative; but all the factors of the first are negative; therefore their number must be odd, which proves the proposition.

In this proposition it is to be understood that multiple roots are counted a number of times equal to the degree of their multiplicity.

It is instructive to apply the graphic method of treatment to the theorem of the present Article. From this point of view it appears almost intuitively true; for it is evident that when any two points are connected by a curve, the portion of the curve between these points must cut the axis an odd number of times when the points are on opposite sides of the axis; and an even number of times, or not at all, when the points are on the same side of the axis.

Examples.

1. If the signs of the terms of an equation be all positive, it cannot have a positive root.

2. If the signs of the terms of any complete equation be alternately positive and negative, it cannot have a negative root.

3. If an equation consist of a number of terms connected by + signs followed by a number of terms connected by − signs, it has one positive root and no more.

Apply Art. 12, substituting 0 and ∞; and Art. 19.

4. If an equation involve only even powers of x, and if all the coefficients have positive signs, it cannot have a real root.

Apply Arts. 19 and 20.

5. If an equation involve only odd powers of x, and if the coefficients have all positive signs, it has the root zero and no other real root.

6. If an equation be complete, the number of continuations of sign in $f(x)$ is the same as the number of variations of sign in $f(-x)$.

7. When an equation is complete; if all its roots be real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of continuations of sign.

8. An equation having an even number of variations of sign must have its last sign positive, and one having an odd number of variations must have its last sign negative.

Take the highest power of x with positive coefficient (see Art. 4).

9. Hence prove that if an equation have an even number of variations it must have an equal or less even number of positive roots; and if it have an odd number of variations it must have an equal or less odd number of positive roots; in other

words, the number of positive roots when less than the number of variations must differ from it by an even number.

Substitute 0 and ∞, and apply Art. 22.

10. Find an inferior limit to the number of imaginary roots of the equation

$$x^6 - 3x^2 - x + 1 = 0.$$

Ans. At least two imaginary roots.

11. Find the nature of the roots of the equation

$$x^4 + 15x^2 + 7x - 11 = 0.$$

Apply Arts. 14, 19, 20.

Ans. One positive, one negative, two imaginary.

12. Show that the equation

$$x^3 + qx + r = 0,$$

where q and r are essentially positive, has one negative and two imaginary roots.

13. Show that the equation

$$x^3 - qx + r = 0,$$

where q and r are essentially positive, has one negative root; and that the other two roots are either imaginary or both positive.

14. Show that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \dots + \frac{L^2}{x-l} = x - m,$$

where $a, b, c, \dots l$ are numbers all different from one another, cannot have an imaginary root.

Substitute $\alpha + \beta\sqrt{-1}$ and $\alpha - \beta\sqrt{-1}$ in succession for x, and subtract. We get an expression which can vanish only on the supposition $\beta = 0$.

15. Show that the equation

$$x^n - 1 = 0$$

has, when n is even, two real roots, 1 and -1, and no other real root; and, when n is odd, the real root 1, and no other real root.

This and the next example follow readily from Arts. 19 and 20.

16. Show that the equation

$$x^n + 1 = 0$$

has, when n is even, no real root; and, when n is odd, the real root -1, and no other real root.

17. Solve the equation

$$x^4 + 2qx^3 + 3q^2x^2 + 2q^3x - r^4 = 0.$$

This is equivalent to

$$(x^2 + qx + q^2)^2 - q^4 - r^4 = 0.$$

$$\textit{Ans.}\ -\frac{1}{2}q + \sqrt{-\frac{3}{4}q^2 + \sqrt{q^4 + r^4}}.$$

The different signs of the radicals give four combinations, and the expression here written involves the four roots.

18. Form the equation which has for roots the different values of the expression

$$2 + \theta\sqrt{7} + \sqrt{11 + \theta\sqrt{7}},$$

where $\theta^2 = 1$.

If no restriction had been made by the introduction of θ, this expression would have 8 values. The $\sqrt{7}$ must now be taken with the same sign where it occurs under the second radical and free from it. There are, therefore, only four values in all. *Ans.* $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$.

19. Form the equation which has for roots the four values of

$$-9 + \theta\sqrt{137} + 3\sqrt{34 - 2\theta\sqrt{137}},$$

where $\theta^2 = 1$. *Ans.* $x^4 + 36x^3 - 400x^2 - 3168x + 7744 = 0$.

20. Form an equation with rational coefficients which shall have for roots all the values of the expression

$$\theta_1\sqrt{p} + \theta_2\sqrt{q} + \theta_3\sqrt{r},$$

where $$\theta_1^2 = 1, \quad \theta_2^2 = 1, \quad \theta_3^2 = 1.$$

There are eight different values of this expression, viz.,

$$\begin{array}{ll}
\sqrt{p} + \sqrt{q} + \sqrt{r}, & -\sqrt{p} - \sqrt{q} - \sqrt{r}, \\
\sqrt{p} - \sqrt{q} - \sqrt{r}, & -\sqrt{p} + \sqrt{q} + \sqrt{r}, \\
-\sqrt{p} + \sqrt{q} - \sqrt{r}, & \sqrt{p} - \sqrt{q} + \sqrt{r}, \\
-\sqrt{p} - \sqrt{q} + \sqrt{r}, & \sqrt{p} + \sqrt{q} - \sqrt{r}.
\end{array}$$

Assume

$$x = \theta_1\sqrt{p} + \theta_2\sqrt{q} + \theta_3\sqrt{r}.$$

Squaring, we have

$$x^2 = p + q + r + 2(\theta_2\theta_3\sqrt{qr} + \theta_3\theta_1\sqrt{rp} + \theta_1\theta_2\sqrt{pq}).$$

Transposing, and squaring again,

$$(x^2 - p - q - r)^2 = 4(qr + rp + pq) + 8\theta_1\theta_2\theta_3\sqrt{pqr}\,(\theta_1\sqrt{p} + \theta_2\sqrt{q} + \theta_3\sqrt{r}).$$

Transposing, substituting x for $\theta_1\sqrt{p} + \theta_2\sqrt{q} + \theta_3\sqrt{r}$, and squaring, we obtain the final equation free from radicals

$$\{x^4 - 2x^2(p + q + r) + p^2 + q^2 + r^2 - 2qr - 2rp - 2pq\}^2 = 64pqrx^2.$$

This is an equation of the eighth degree, whose roots are the values above written. Since θ_1, θ_2, θ_3 have disappeared, it is indifferent which of the eight roots $\pm\sqrt{p} \pm \sqrt{q} \pm \sqrt{r}$ is assumed equal to x in the first instance. The final equation is that which would have been obtained if each of the 8 roots had been subtracted from x, and the continued product formed, as in Ex. 6, Art. 16.

CHAPTER III.

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF EQUATIONS, WITH APPLICATIONS TO SYMMETRIC FUNCTIONS OF THE ROOTS.

23. **Relations between the Roots and Coefficients.**— Taking for simplicity the coefficient of the highest power of x as unity, and representing, as in Art. 16, the n roots of an equation by $a_1, a_2, a_3, \ldots a_n$, we have the following identity :—

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n$$
$$\equiv (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n). \qquad (1)$$

When the factors of the second member of this identity are multiplied together, the highest power of x in the product is x^n; the coefficient of x^{n-1} is the sum of the n quantities $-a_1$, $-a_2$, &c., viz. the roots with their signs changed; the coefficient of x^{n-2} is the sum of the products of these quantities taken two by two; the coefficient of x^{n-3} is the sum of their products taken three by three; and so on, the last term being the product of all the roots with their signs changed. Equating, therefore, the coefficients of x on each side of the identity (1), we have the following series of equations :—

$$\left.\begin{aligned}
p_1 &= -\,(a_1 + a_2 + a_3 + \ldots + a_n), \\
p_2 &= \quad (a_1 a_2 + a_1 a_3 + a_2 a_3 + \ldots + a_{n-1} a_n), \\
p_3 &= -\,(a_1 a_2 a_3 + a_1 a_3 a_4 + \ldots + a_{n-2} a_{n-1} a_n), \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
p_n &= (-1)^n a_1 a_2 a_3 \ldots a_{n-1} a_n,
\end{aligned}\right\} \qquad (2)$$

which enable us to state the relations between the roots and coefficients as follows :—

Theorem.—*In every algebraic equation, the coefficient of whose highest term is unity, the coefficient p_1 of the second term with its sign changed is equal to the sum of the roots.*

The coefficient p_2 of the third term is equal to the sum of the products of the roots taken two by two.

The coefficient p_3 of the fourth term with its sign changed is equal to the sum of the products of the roots taken three by three; and so on, the signs of the coefficients being taken alternately negative and positive, and the number of roots multiplied together in each term of the corresponding function of the roots increasing by unity, till finally that function is reached which consists of the product of the n roots.

When the coefficient a_0 of x^n is not unity (see Art. 1), we must divide each term of the equation by it. The sum of the roots is then equal to $-\frac{a_1}{a_0}$; the sum of their products in pairs is equal to $\frac{a_2}{a_0}$; and so on.

Cor. 1.—Every root of an equation is a divisor of the absolute term of the equation.

Cor. 2.—If the roots of an equation be all positive, the coefficients (including that of the highest power of x) will be alternately positive and negative; and if the roots be all negative, the coefficients will be all positive. This is obvious from the equations (2) [cf. Arts. 19 and 20].

24. **Applications of the Theorem.**—Since the equations (2) of the preceding Article supply n distinct relations between the n roots and the coefficients, it might perhaps be supposed that some advantage is thereby gained in the general solution of the equation. Such, however, is not the case; for suppose it were attempted to determine by means of these equations a root, a_1, of the original equation, this could be effected only by the elimination of the other roots by means of the given equations, and the consequent determination of a final equation of which

a_1 is one of the roots. Now, in whatever way this final equation is obtained, it must have for solution not only a_1, but each of the other roots $a_2, a_3, \ldots a_n$; for, since all the roots enter in the same manner in the equations (2), if it had been proposed to determine a_2 (or any other root) by the elimination of the rest, our final equation could differ from that obtained for a_1 only by the substitution of a_2 (or that other root) for a_1. The final equation arrived at, therefore, by the process of elimination must have the n quantities $a_1, a_2, \ldots \ldots a_n$ for roots; and cannot, consequently, be easier of solution than the given equation. This final equation is, in fact, the original equation itself, with the root we are seeking substituted for x. This we shall show for the particular case of a cubic. The process here employed is general, and may be applied to an equation of any degree. Let a, β, γ be the roots of the equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0.$$

We have, by Art. 23,

$$p_1 = -(a + \beta + \gamma),$$
$$p_2 = \quad a\beta + a\gamma + \beta\gamma,$$
$$p_3 = -a\beta\gamma.$$

Multiplying the first of these equations by a^2, the second by a, and adding the three, we find

$$p_1a^2 + p_2a + p_3 = -a^3,$$

or

$$a^3 + p_1a^2 + p_2a + p_3 = 0,$$

which is the given cubic with a in the place of x.

The student can take as an exercise to prove the same result in the case of an equation of the fourth degree. In the corresponding treatment of the general case the successive equations of Art. 23 are to be multiplied by $a^{n-1}, a^{n-2}, a^{n-3}$, &c., and added.

Although the equations (2) afford, as we have just seen, no assistance in the general solution of the equation, they are often of use in facilitating the solution of numerical equations when any particular relations among the roots are known to exist. They may also be employed to establish the relations which

must obtain among the coefficients of algebraical equations corresponding to known relations among the roots.

Examples.

1. Solve the equation

$$x^3 - 5x^2 - 16x + 80 = 0,$$

the sum of two of its roots being equal to zero.

Let the roots be α, β, γ. We have then

$$\alpha + \beta + \gamma = \quad 5,$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -16,$$

$$\alpha\beta\gamma = -80.$$

Taking $\beta + \gamma = 0$, we have, from the first of these, $\alpha = 5$; and from either the second or third we obtain $\beta\gamma = -16$. We find for β and γ the values 4 and -4. Thus the three roots are 5, 4, -4.

2. Solve the equation

$$x^3 - 3x^2 + 4 = 0,$$

two of its roots being equal.

Let the three roots be α, α, β. We have

$$2\alpha + \beta = 3,$$

$$\alpha^2 + 2\alpha\beta = 0,$$

from which we find $\alpha = 2$, and $\beta = -1$. The roots are 2, 2, -1.

3. The equation

$$x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$$

has two pairs of equal roots; find them.

Let the roots be α, α, β, β; we have, therefore,

$$2\alpha + 2\beta = -4,$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = -2,$$

from which we obtain for α and β the values 1 and -3.

4. Solve the equation

$$x^3 - 9x^2 + 14x + 24 = 0,$$

two of whose roots are in the ratio of 3 to 2.

Let the roots be α, β, γ, with the relation $2\alpha = 3\beta$. By elimination of α we easily obtain

$$5\beta + 2\gamma = 18,$$

$$3\beta^2 + 5\beta\gamma = 28,$$

from which we have the following quadratic for β :—

$$19\beta^2 - 90\beta + 56 = 0.$$

The roots of this are 4, and $\frac{14}{19}$; the former gives for α and γ the values 6 and -1. The three roots are 6, 4, -1. The student will here ask what is the significance of the value $\frac{14}{19}$ of β; and the same difficulty may have presented itself in the previous examples. It will be observed that in examples of this nature we never require all the relations between the roots and coefficients in order to determine the required unknown quantities. The reason of this is, that the given condition establishes one or more relations amongst the roots. Whenever the equations employed appear to furnish more than one system of values for the roots, the actual roots are easily determined by the condition that they must satisfy the equation (or equations) between the roots and coefficients which we have not made use of in determining them. Thus, in the present example, the value $\beta = 4$ gives a system satisfying the omitted equation

$$\alpha\beta\gamma = -24;$$

while the value $\beta = \frac{14}{19}$ gives a system not satisfying this equation, and is therefore to be rejected.

5. Solve the equation

$$x^3 - 9x^2 + 23x - 15 = 0,$$

whose roots are in arithmetical progression.

Let the roots be $\alpha - \delta$, α, $\alpha + \delta$; we have at once

$$3\alpha = 9,$$

$$3\alpha^2 - \delta^2 = 23,$$

from which we obtain the three roots 1, 3, 5.

6. Solve the equation

$$x^4 + 2x^3 - 21x^2 - 22x + 40 = 0,$$

whose roots are in arithmetical progression.

Assume for the roots $\alpha - 3\delta$, $\alpha - \delta$, $\alpha + \delta$, $\alpha + 3\delta$.

Ans. -5, -2, 1, 4

7. Solve the equation

$$27x^3 + 42x^2 - 28x - 8 = 0,$$

whose roots are in geometric progression.

Assume for the roots $\alpha\rho$, α, $\frac{\alpha}{\rho}$. From the third of the equations (2), Art. 23, we have $\alpha^3 = \frac{8}{27}$, or $\alpha = \frac{2}{3}$. Either of the remaining two equations gives a quadratic for ρ.

Ans. -2, $\frac{2}{3}$, $\frac{-2}{9}$.

8. Solve the equation

$$3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0,$$

whose roots are in geometric progression.

Assume for the roots $\frac{a}{\rho^3}$, $\frac{a}{\rho}$, $a\rho$, $a\rho^3$. Employ the second and fourth of the equations (2), Art. 23.

Ans. $\frac{1}{3}$, 1, 3, 9.

9. Solve the equation

$$x^4 + 15x^3 + 70x^2 + 120x + 64 = 0,$$

whose roots are in geometric progression. *Ans.* $-1, -2, -4, -8$.

10. Solve the equation

$$6x^3 - 11x^2 + 6x - 1 = 0,$$

whose roots are in harmonic progression.

Take the roots to be α, β, γ. We have here the relation

$$\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta};$$

hence

$$\beta\gamma + \gamma\alpha + \alpha\beta = 3\gamma\alpha; \text{ \&c.}$$

Ans. 1, $\frac{1}{2}$, $\frac{1}{3}$.

11. Solve the equation

$$81x^3 - 18x^2 - 36x + 8 = 0,$$

whose roots are in harmonic progression.

Ans. $\frac{2}{9}$, $\frac{2}{3}$, $-\frac{2}{3}$.

12. If the roots of the equation

$$x^3 - px^2 + qx - r = 0$$

be in harmonic progression, show that the mean root is $\frac{3r}{q}$.

13. The equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$$

has two roots equal in magnitude and opposite in sign; determine all the roots.

Take $\alpha + \beta = 0$, and employ the first and third of equations (2), Art. 23.

Ans. $\sqrt{3}$, $-\sqrt{3}$, $1 \pm \sqrt{-6}$

14. The equation

$$3x^4 - 25x^3 + 50x^2 - 50x + 12 = 0$$

has two roots whose product is 2; find all the roots.

Ans. 6, $\frac{1}{3}$, $1 \pm \sqrt{-1}$.

15. One of the roots of the cubic

$$x^3 - px^2 + qx - r = 0$$

is double another; show that it may be found from a quadratic equation.

16. Show that all the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n = 0$$

can be obtained when they are in arithmetical progression.

Let the roots be α, $\alpha + \delta$, $\alpha + 2\delta$, $\ldots$ $\alpha + (n-1)\delta$. The first of equations (2) gives

$$-p_1 = n\alpha + \{1 + 2 + 3 + \ldots + (n-1)\}\,\delta$$

$$= n\alpha + \frac{n(n-1)}{2}\,\delta. \qquad (1)$$

Again, since the sum of the squares of any number of quantities is equal to the square of their sum minus twice the sum of their products in pairs, we have the equation

$$p_1^2 - 2p_2 = \alpha^2 + (\alpha+\delta)^2 + (\alpha + 2\delta)^2 + \ldots$$

$$= n\alpha^2 + n(n-1)\,\alpha\delta + \frac{n(n-1)(2n-1)}{6}\,\delta^2. \qquad (2)$$

Subtracting the square of (1) from n times the equation (2), we find δ^2 in terms of p_1 and p_2. We can then find α from equation (1). Thus all the roots can be expressed in terms of the coefficients p_1 and p_2.

17. Find the condition which must be satisfied by the coefficients of the equation

$$x^3 - px^2 + qx - r = 0,$$

when two of its roots α, β are connected by a relation $\alpha + \beta = 0$.

Ans. $pq - r = 0$.

18. Find the condition that the cubic

$$x^3 - px^2 + qx - r = 0$$

should have its roots in geometric progression. *Ans.* $p^3 r - q^3 = 0$

19. Find the condition that the same cubic should have its roots in harmonic progression (see Ex. 12). *Ans.* $27r^2 - 9pqr + 2q^3 = 0$.

20. Find the condition that the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

should have two roots connected by the relation $\alpha + \beta = 0$; and determine in that case two quadratic equations which shall have for roots (1) α, β; and (2) γ, δ.

$$\textit{Ans.}\ pqr - p^2 s - r^2 = 0, \quad (1)\ px^2 + r = 0, \quad (2)\ x^2 + px + \frac{ps}{r} = 0.$$

21. Find the condition that the biquadratic of Ex. 20 should have its roots connected by the relation $\beta + \gamma = \alpha + \delta$. *Ans.* $p^3 - 4pq + 8r = 0$.

22. Find the condition that the roots α, β, γ, δ of

$$x^4 + px^3 + qx^2 + rx + s = 0$$

should be connected by the relation $\alpha\beta = \gamma\delta$. *Ans.* $p^2 s - r^2 = 0$.

23. Show that the condition obtained in Ex. 22 is satisfied when the roots of the biquadratic are in geometric progression.

25. **Depression of an Equation when a relation exists between two of its Roots.**—The examples given in the preceding Article illustrate the use of the equations connecting the roots and coefficients in determining the roots in particular cases when known relations exist among them. We shall now show in general, that *if a relation of the form $\beta = \phi(\alpha)$ exist between two of the roots of an equation $f(x) = 0$, the equation may be depressed two dimensions.*

Let $\phi(x)$ be substituted for x in the identity

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

then $f(\phi(x)) \equiv a_0(\phi(x))^n + a_1(\phi(x))^{n-1} + \ldots + a_{n-1}\phi(x) + a_n.$

We represent, for convenience, the second member of this identity by $F(x)$. Substituting α for x, we have

$$F(\alpha) \equiv f(\phi(\alpha)) \equiv f(\beta) = 0;$$

hence α satisfies the equation $F(x) = 0$, and it also satisfies the equation $f(x) = 0$; hence the polynomials $f(x)$ and $F(x)$ have a common measure $x - \alpha$; thus α can be determined, and from it $\phi(\alpha)$ or β, and the given equation can be depressed two dimensions.

Examples.

1. The equation

$$x^3 - 5x^2 - 4x + 20 = 0$$

has two roots whose difference $= 3$: find them.

Here $\beta - \alpha = 3$, $\beta = 3 + \alpha$; substitute $x + 3$ for x in the given polynomial $f(x)$; it becomes $x^3 + 4x^2 - 7x - 10$; the common measure of this and $f(x)$ is $x - 2$; from which $\alpha = 2$, $\beta = 5$; the third root is -2.

2. The equation

$$x^4 - 5x^3 + 11x^2 - 13x + 6 = 0$$

has two roots connected by the relation $2\beta + 3\alpha = 7$: find all the roots.

Ans. $1,\ 2,\ 1 \pm \sqrt{-2}$.

It may be observed here, that when two polynomials $f(x)$ and $F(x)$ have common factors, these factors may be obtained by the ordinary process of finding the common measure. Thus,

if we know that two given equations have common roots, we can obtain these roots by equating to zero the greatest common measure of the given polynomials.

EXAMPLES.

1. The equations

$$2x^3+5x^2-\ \ 6x-\ \ 9=0,$$

$$3x^3+7x^2-11x-15=0$$

have two common roots: find them. *Ans.* $-1,\ \ -3.$

2. The equations

$$x^3+px^2+qx+r=0,$$

$$x^3+p'x^2+q'x+r'=0$$

have two common roots: find the quadratic whose roots are these two, and find also the third root of each.

Ans. $x^2+\dfrac{q-q'}{p-p'}x+\dfrac{r-r'}{p-p'}=0,\quad \dfrac{-r(p-p')}{r-r'},\quad \dfrac{-r'(p-p')}{r-r'}.$

26. **The Cube Roots of Unity.** — Equations of the forms

$$x^n-1=0,\quad x^n+1=0,$$

consisting of the highest and absolute terms only, are called *binomial equations.* The roots of the former are called the *n n*th *roots of unity.* A general discussion of these forms will be given in a subsequent chapter. We confine ourselves at present to the simple case of the binomial cubic, for which certain useful properties of the roots can be easily established. It has been already shown (see Ex. 5, Art. 16), that the roots of the cubic

$$x^3-1=0$$

are

$$1,\ -\frac{1}{2}+\frac{1}{2}\sqrt{-3},\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

If either of the imaginary roots be represented by ω, the other is easily seen to be ω^2, by actually squaring; or we may see the same thing as follows:—If ω be a root of the cubic, ω^2 must also be a root; for, since $\omega^3=1$, we get, by squaring,

$\omega^6 = 1$, or $(\omega^2)^3 = 1$, thus showing that ω^2 satisfies the cubic $x^3 - 1 = 0$. We have then the identity

$$x^3 - 1 \equiv (x-1)(x-\omega)(x-\omega^2).$$

Changing x into $-x$, we get the following identity also:–

$$x^3 + 1 \equiv (x+1)(x+\omega)(x+\omega^2),$$

which furnishes the roots of

$$x^3 + 1 = 0.$$

Whenever in any product of quantities involving the imaginary cube roots of unity any power higher than the second presents itself, it can be replaced by ω, or ω^2, or by unity; for example,

$$\omega^4 = \omega^3 . \omega = \omega, \quad \omega^5 = \omega^3 . \omega^2 = \omega^2, \quad \omega^6 = \omega^3 . \omega^3 = 1, \text{ \&c.}$$

The first or second of equations (2), Art. 23, gives the following property of the imaginary cube roots:—

$$1 + \omega + \omega^2 = 0.$$

By the aid of this equation any expression involving real quantities and the imaginary cube roots can be written in any of the forms $P + \omega Q$, $P + \omega^2 Q$, $\omega P + \omega^2 Q$.

Examples.

1. Show that the product

$$(\omega m + \omega^2 n)(\omega^2 m + \omega n)$$

is rational. *Ans.* $m^2 - mn + n^2$

2. Prove the following identities:—

$$m^3 + n^3 \equiv (m+n)(\omega m + \omega^2 n)(\omega^2 m + \omega n),$$

$$m^3 - n^3 \equiv (m-n)(\omega m - \omega^2 n)(\omega^2 m - \omega n).$$

3. Show that the product

$$(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)$$

is rational.

Ans. $\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta$.

4. Prove the identity

$$(\alpha+\beta+\gamma)(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) \equiv \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma.$$

5. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 + (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta).$$

Apply Ex. 2.

6. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 - (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv -3\sqrt{-3}\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Apply Ex. 2, and substitute for $\omega - \omega^2$ its value $\sqrt{-3}$.

7. Prove the identity

$$\alpha'^3 + \beta'^3 + \gamma'^3 - 3\alpha'\beta'\gamma' \equiv (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma)^2,$$

where

$$\alpha' \equiv \alpha^2 + 2\beta\gamma, \quad \beta' \equiv \beta^2 + 2\gamma\alpha, \quad \gamma' \equiv \gamma^2 + 2\alpha\beta.$$

8. Form the equation whose roots are

$$m+n, \quad \omega m + \omega^2 n, \quad \omega^2 m + \omega n.$$

Ans. $x^3 - 3mnx - (m^3 + n^3) = 0.$

9. Form the equation whose roots are

$$l+m+n, \quad l + \omega m + \omega^2 n, \quad l + \omega^2 m + \omega n.$$

Ans. $x^3 - 3lx^2 + 3(l^2 - mn)x - (l^3 + m^3 + n^3 - 3lmn) = 0.$

It is important to observe that, corresponding to the n n^{th} roots of unity, there are n n^{th} roots of any quantity. The roots of the equation

$$x^n - a = 0$$

are the n n^{th} roots of a.

The three cube roots, for example, of a are

$$\sqrt[3]{a}, \quad \omega\sqrt[3]{a}, \quad \omega^2\sqrt[3]{a},$$

where $\sqrt[3]{a}$ represents the real cube root according to the ordinary arithmetical interpretation. Each of these values satisfies the cubic equation $x^3 - a = 0$. It is to be observed that the three cube roots may be obtained by multiplying *any one* of the three above written by 1, ω, ω^2.

In addition, therefore, to the real cube root there are two imaginary cube roots obtained by multiplying the real cube root by the imaginary cube roots of unity. Thus, besides the ordinary cube root 3, the number 27 has the two imaginary cube roots

$$-\frac{3}{2} + \frac{3}{2}\sqrt{-3}, \quad -\frac{3}{2} - \frac{3}{2}\sqrt{-3},$$

as the student can easily verify by actual cubing.

10. Form a rational equation which shall have

$$\omega\sqrt[3]{Q + \sqrt{Q^2 + P^3}} + \omega^2\sqrt[3]{Q - \sqrt{Q^2 + P^3}}$$

for a root; where $\omega^3 = 1$.

Compare Ex. 8. *Ans.* $x^3 + 3Px - 2Q = 0.$

11. Form an equation with rational coefficients which shall have

$$\theta_1\sqrt[3]{P}+\theta_2\sqrt[3]{Q}$$

for a root, where $\theta_1^3=1$, and $\theta_2^3=1$.

Cubing both sides of the equation

$$x=\theta_1\sqrt[3]{P}+\theta_2\sqrt[3]{Q},$$

and substituting x for its value on the right-hand side, we get

$$x^3-P-Q=3\theta_1\theta_2\sqrt[3]{PQ}\,.\,x.$$

Cubing again, we have

$$(x^3-P-Q)^3=27PQx^3.$$

Since θ_1 and θ_2 may each have any one of the values 1, ω, ω^2, the nine roots of this equation are

$$\begin{array}{ccc}
\sqrt[3]{P}+\sqrt[3]{Q}, & \omega\sqrt[3]{P}+\omega\sqrt[3]{Q}, & \omega^2\sqrt[3]{P}+\omega^2\sqrt[3]{Q},\\
\omega\sqrt[3]{P}+\omega^2\sqrt[3]{Q}, & \omega^2\sqrt[3]{P}+\sqrt[3]{Q}, & \omega\sqrt[3]{P}+\sqrt[3]{Q},\\
\omega^2\sqrt[3]{P}+\omega\sqrt[3]{Q}, & \sqrt[3]{P}+\omega^2\sqrt[3]{Q}, & \sqrt[3]{P}+\omega\sqrt[3]{Q}.
\end{array}$$

We see also that, since θ_1 and θ_2 have disappeared from the final equation, it is indifferent which of these nine roots is assumed equal to x in the first instance. The resulting equation is that which would have been obtained by multiplying together the nine factors of the form $x-\sqrt[3]{P}-\sqrt[3]{Q}$ obtained from the nine roots above written.

12. Form separately the three cubic equations whose roots are the groups in three (written in vertical columns in Ex. 11) of the roots of the equation of the preceding example.

We can write these down from Ex. 8, taking first m and n equal to $\sqrt[3]{P}$, $\sqrt[3]{Q}$; then equal to $\omega\sqrt[3]{P}$, $\omega\sqrt[3]{Q}$; and finally equal to $\omega^2\sqrt[3]{P}$, $\omega^2\sqrt[3]{Q}$.

$$\textit{Ans.}\quad x^3-3\sqrt[3]{PQ}x-P-Q=0,$$
$$x^3-3\omega^2\sqrt[3]{PQ}x-P-Q=0,$$
$$x^3-3\omega\sqrt[3]{PQ}x-P-Q=0.$$

27. **Symmetric Functions of the Roots.**—Symmetric functions of the roots of an equation are those functions in which all the roots are alike involved, so that the expression is unaltered in value when any two of the roots are interchanged. For example, the functions of the roots (the sum, the sum of the products in pairs, &c.) with which we were concerned in Art. 23 are of this nature; for, as the student will readily perceive, if in any of these expressions the root a_1, let us say, be written in

every place where a_2 occurs, and a_2 in every place where a_1 occurs, the value of the expression will be unchanged.

The functions discussed in Art. 23 are the simplest symmetric functions of the roots, each root entering in the first degree only in any term of any one of them.

We can, without knowing the values of the roots separately in terms of the coefficients, obtain by means of the equations (2) of Art. 23 the values in terms of the coefficients of an infinite variety of symmetric functions of the roots. It will be shown in a subsequent chapter, when the discussion of this subject is resumed, that any rational symmetric function whatever of the roots can be so expressed. The examples appended to this Article, most of which have reference to the simple cases of the cubic and biquadratic, are sufficient for the present to illustrate the usual elementary methods of obtaining such expressions in terms of the coefficients.

It is usual to represent a symmetric function by the Greek letter Σ attached to one term of it, from which the entire expression may be written down. Thus, if α, β, γ be the roots of a cubic, $\Sigma\alpha^2\beta^2$ represents the symmetric function

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2,$$

where all possible products in pairs are taken, and all the terms added after each is separately squared. Again, in the same case, $\Sigma\alpha^2\beta$ represents the sum

$$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta,$$

where all possible permutations of the roots two by two are taken, and the first root in each term then squared.

As an illustration in the case of a biquadratic we take $\Sigma\alpha^2\beta^2$, whose expanded form is as follows:—

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2.$$

By the aid of the various symmetric functions which occur among the following examples the student will acquire a facility in writing out in all similar cases the entire expression when the typical term is given.

EXAMPLES.

1. Find the value of $\Sigma\alpha^2\beta$ of the roots of the cubic equation

$$x^3 + px^2 + qx + r = 0.$$

Multiplying together the equations

$$\alpha + \beta + \gamma = -p,$$
$$\beta\gamma + \gamma\alpha + \alpha\beta = \quad q,$$

we obtain $$\Sigma\alpha^2\beta + 3\alpha\beta\gamma = -pq;$$

hence $$\Sigma\alpha^2\beta = 3r - pq.$$

2. Find for the same cubic the value of

$$\alpha^2 + \beta^2 + \gamma^2. \qquad \textit{Ans. } \Sigma\alpha^2 = p^2 - 2q.$$

3. Find for the same cubic the value of

$$\alpha^3 + \beta^3 + \gamma^3.$$

Multiplying the values of $\Sigma\alpha$ and $\Sigma\alpha^2$, we obtain

$$\alpha^3 + \beta^3 + \gamma^3 + \Sigma\alpha^2\beta = -p^3 + 2pq;$$

hence, by Ex. 1,

$$\Sigma\alpha^3 = -p^3 + 3pq - 3r.$$

4. Find for the same cubic the value of

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2.$$

We easily obtain

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma) = q^2,$$

from which $$\Sigma\alpha^2\beta^2 = q^2 - 2pr.$$

5. Find for the same cubic the value of

$$(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta).$$

This is equal to

$$2\alpha\beta\gamma + \Sigma\alpha^2\beta. \qquad \textit{Ans. } r - pq.$$

6. Find the value of the symmetric function

$$\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta$$
$$+ \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$$

of the roots of the biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Multiplying together

$$\alpha + \beta + \gamma + \delta = -p,$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r,$$

we obtain $$\Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta = pr;$$

hence $$\Sigma\alpha^2\beta\gamma = pr - 4s$$

7. Find for the same biquadratic the value of the symmetric function

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$

Squaring $\Sigma\alpha$, we easily obtain

$$\Sigma\alpha^2 = p^2 - 2q.$$

8. Find for the same biquadratic the value of the symmetric function

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2.$$

Squaring the equation

$$\Sigma\alpha\beta = q,$$

we obtain

$$\Sigma\alpha^2\beta^2 + 2\Sigma\alpha^2\beta\gamma + 6\alpha\beta\gamma\delta = q^2;$$

hence, by Ex. 6,

$$\Sigma\alpha^2\beta^2 = q^2 - 2pr + 2s.$$

9. Find for the same biquadratic the value of $\Sigma\alpha^3\beta$.

To form this symmetric function we take the two permutations $\alpha\beta$ and $\beta\alpha$ of the letters α, β; these give two terms $\alpha^3\beta$ and $\beta^3\alpha$ of Σ. We have similarly two terms from every other pair of the letters α, β, γ, δ; so that the symmetric function consists of 12 terms in all.

Multiply together the two equations

$$\Sigma\alpha\beta = q, \quad \Sigma\alpha^2 = p^2 - 2q;$$

and observe that

$$\Sigma\alpha^2 \, \Sigma\alpha\beta \equiv \Sigma\alpha^3\beta + \Sigma\alpha^2\beta\gamma.$$

[It is convenient to remark here, that results of the kind expressed by this last equation can be verified by the consideration that the number of terms in both members of the equation must be the same. Thus, in the present instance, since $\Sigma\alpha^2$ contains 4 terms, and $\Sigma\alpha\beta$ 6 terms, their product must contain 24; and these are in fact the 12 terms which form $\Sigma\alpha^3\beta$, together with the 12 which form $\Sigma\alpha^2\beta\gamma$.]

Using the results of previous examples, we have, therefore,

$$\Sigma\alpha^3\beta = p^2q - 2q^2 - pr + 4s.$$

10. Find for the same biquadratic the value of

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4.$$

Squaring $\Sigma\alpha^2$, and employing results already obtained,

$$\Sigma\alpha^4 = p^4 - 4p^2q + 2q^2 + 4pr - 4s.$$

11. Find the value, in terms of the coefficients, of the sum of the squares of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0.$$

Squaring $\Sigma\alpha_1$, we easily find

$$p_1^2 = \Sigma\alpha_1^2 + 2\Sigma\alpha_1\alpha_2;$$

hence

$$\Sigma\alpha_1^2 = p_1^2 - 2p_2.$$

12. Find the value, in terms of the coefficients, of the sum of the reciprocals of the roots of the equation in the preceding example.

From the second last, and last of the equations of Art. 23, we have

$$\alpha_2\alpha_3 \ldots . \alpha_n + \alpha_1\alpha_3 \ldots . \alpha_n + \ldots . + \alpha_1\alpha_2 \ldots . \alpha_{n-1} = (-1)^{n-1}p_{n-1},$$

$$\alpha_1\alpha_2\alpha_3 \ldots . \alpha_n = (-1)^n p_n;$$

dividing the former by the latter, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \ldots . + \frac{1}{\alpha_n} = \frac{-p_{n-1}}{p_n},$$

or

$$\Sigma \frac{1}{\alpha_1} = \frac{-p_{n-1}}{p_n}.$$

In a similar manner the sum of the products in pairs, in threes, &c., of the reciprocals of the roots can be found by dividing the 3rd last, or 4th last, &c., coefficient by the last.

13. Find for the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

the value, in terms of the coefficients, of the following symmetric function of the roots α, β, γ:—

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2.$$

N.B.—It will often be found convenient to write, as in the present example, an equation with *binomial coefficients*, that is, numerical coefficients the same as those which occur in the expansion by the binomial theorem, in addition to the literal coefficients a_0, a_1, &c. Here the equation being of the third degree, the successive numerical coefficients are those which occur in the expansion to the third power, viz. 1, 3, 3, 1.

We easily obtain

$$a_0^2\{(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2\} = 18(a_1^2 - a_0a_2).$$

14. Express in terms of the coefficients of the cubic in the preceding example the successive coefficients of the quadratic

$$(x-\alpha)^2(\beta-\gamma)^2 + (x-\beta)^2(\gamma-\alpha)^2 + (x-\gamma)^2(\alpha-\beta)^2 = 0,$$

where α, β, γ are the roots of the cubic.

Here, in addition to the symmetric function of the preceding example, we have to calculate also the two following:—

$$\alpha(\beta-\gamma)^2 + \beta(\gamma-\alpha)^2 + \gamma(\alpha-\beta)^2,$$

$$\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2.$$

Ans. $(a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)x + (a_1a_3 - a_2^2) = 0.$

15. Find for the cubic of Example 13 the value in terms of the coefficients of

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

Since $$2\alpha-\beta-\gamma = 3\alpha-(\alpha+\beta+\gamma) = 3\alpha + \frac{3a_1}{a_0},$$

the required value is easily obtained by substituting $-\frac{a_1}{a_0}$ for x in the identity

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 \equiv a_0 (x-\alpha)(x-\beta)(x-\gamma).$$

Ans. $a_0^3 (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) = -27 (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3)$.

16. Find, in terms of the coefficients of the biquadratic equation

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0,$$

the value of the following symmetric function of the roots :—

$$(\beta-\gamma)^2 (\alpha-\delta)^2 + (\gamma-\alpha)^2 (\beta-\delta)^2 + (\alpha-\beta)^2 (\gamma-\delta)^2.$$

Here the equation is written with numerical coefficients corresponding to the expansion of the binomial to the 4th power. The symmetric function in question is easily seen to be identical with

$$2\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma + 12\alpha\beta\gamma\delta.$$

Employing the results of examples 6 and 8, we find

$$a_0^2 \{(\beta-\gamma)^2 (\alpha-\delta)^2 + (\gamma-\alpha)^2 (\beta-\delta)^2 + (\alpha-\beta)^2 (\gamma-\delta)^2\} = 24 (a_0 a_4 - 4a_1 a_3 + 3a_2^2).$$

17. Taking the six products in pairs of the four roots of the equation of Ex. 16, and adding each product, *e.g.* $\alpha\beta$, to that which contains the remaining two roots, $\gamma\delta$, we have the three sums in pairs

$$\beta\gamma + \alpha\delta, \quad \gamma\alpha + \beta\delta, \quad \alpha\beta + \gamma\delta;$$

it is required to find the values in terms of the coefficients of the two following symmetric functions of the roots :—

$$(\gamma\alpha + \beta\delta)(\alpha\beta + \gamma\delta) + (\alpha\beta + \gamma\delta)(\beta\gamma + \alpha\delta) + (\beta\gamma + \alpha\delta)(\gamma\alpha + \beta\delta),$$

$$(\beta\gamma + \alpha\delta)(\gamma\alpha + \beta\delta)(\alpha\beta + \gamma\delta).$$

The former of these is the sum of the products in pairs, and the latter the continued product, of the three expressions above given. As these three functions of the roots are important in the theory of the biquadratic, we shall represent them uniformly by the letters λ, μ, ν. We have, therefore, to find expressions in terms of the coefficients for $\mu\nu + \nu\lambda + \lambda\mu$, and $\lambda\mu\nu$.

The former is $\Sigma\alpha^2\beta\gamma$, and is easily expressed as follows (cf. Ex. 6) :—

$$a_0^2 \Sigma\mu\nu = 4 (4a_1 a_3 - a_0 a_4).$$

The latter is, when multiplied out, equal to

$$\alpha\beta\gamma\delta (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + \alpha^2\beta^2\gamma^2\delta^2 \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right),$$

and we obtain after easy calculations the following :—

$$a_0^3 \lambda\mu\nu = 8 (2a_0 a_3^2 - 3a_0 a_2 a_4 + 2a_1^2 a_4).$$

18. Find, in terms of the coefficients of the biquadratic of Ex. 16, the value of the following symmetric function of the roots :—

$$\{(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)\}\{(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)\}$$
$$\{(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)\}.$$

This is also an important symmetric function in the theory of the biquadratic. To prevent any ambiguity in writing this, or corresponding functions in which the differences of the roots of the biquadratic enter, we explain the notation which will be uniformly employed in this work.

Taking in circular order the three roots α, β, γ, we have the three differences $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$; and subtracting δ from each root in turn, we have the three other differences $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$. We combine these in pairs as follows :—

$$(\beta-\gamma)(\alpha-\delta), \quad (\gamma-\alpha)(\beta-\delta), \quad (\alpha-\beta)(\gamma-\delta).$$

The symmetric function in question is the product of the differences of these three taken as usual in circular order.

Employing the values of λ, μ, ν, in the preceding example, we have

$$-\mu+\nu\equiv(\beta-\gamma)(\alpha-\delta), \quad -\nu+\lambda\equiv(\gamma-\alpha)(\beta-\delta), \quad -\lambda+\mu\equiv(\alpha-\beta)(\gamma-\delta).$$

We have, therefore, to find the value of

$$(2\lambda-\mu-\nu)(2\mu-\nu-\lambda)(2\nu-\lambda-\mu),$$

or

$$(3\lambda-\Sigma\alpha\beta)(3\mu-\Sigma\alpha\beta)(3\nu-\Sigma\alpha\beta),$$

in terms of the coefficients of the biquadratic.

Multiplying this out, substituting the value of $\Sigma\alpha\beta$, and attending to the results of Ex. 17, we obtain the required expression as follows :—

$$a_0^3(2\lambda-\mu-\nu)(2\mu-\nu-\lambda)(2\nu-\lambda-\mu)=-432\{a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3\}.$$

The function of the coefficients here arrived at, as well as those before obtained in Examples 13, 15, and 16, will be found to be of great importance in the theory of the cubic and biquadratic equations.

19. Find, in terms of the coefficients of the biquadratic of Ex. 16, the value of the symmetric function

$$(\alpha-\beta)^2+(\alpha-\gamma)^2+(\alpha-\delta)^2+(\beta-\gamma)^2+(\beta-\delta)^2+(\gamma-\delta)^2.$$

This may be represented briefly by $\Sigma(\alpha-\beta)^2$.

Ans. $a_0^2\Sigma(\alpha-\beta)^2=48(a_1^2-a_0a_2)$.

20. Prove the following relation between the roots and coefficients of the biquadratic of Ex. 16 :—

$$a_0^3(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)=32(a_0^2a_3-3a_0a_1a_2+2a_1^3).$$

28. **Theorems relating to Symmetric Functions.**—The following two theorems, with which we close for the present the discussion of this subject, will be found useful in many instances in verifying the results of the calculation of the values of symmetric functions in terms of the coefficients.

(1). *The sum of the exponents of all the roots in any term of any symmetric function of the roots is equal to the sum of the suffixes in each term of the corresponding value in terms of the coefficients.*

The sum of the exponents is of course the same for every term of the symmetric function, and may be called the *degree in all the roots* of that function. The truth of the theorem will be observed in the particular cases of the Examples 13, 15, 16, 17, &c. of the last Article; and that it must be true in general appears from the equations (2) of Art. 23, for the suffix of each coefficient in those equations is equal to the degree in the roots of the corresponding function of the roots; hence in any product of any powers of the coefficients the sum of the suffixes must be equal to the degree in all the roots of the corresponding function of the roots.

(2). *When an equation is written with binomial coefficients, the expression in terms of the coefficients for any symmetric function of the roots, which is a function of their differences only, is such that the algebraic sum of the numerical factors of all the terms in it is equal to zero.*

The truth of this proposition appears by supposing all the coefficients a_0, a_1, a_2, &c. to become equal to unity in the general equation written with binomial coefficients, viz.,

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + a_n = 0;$$

for the equation then becomes $(x+1)^n = 0$, *i.e.* all the roots become equal; hence any function of the differences of the roots must in that case vanish, and therefore also the function of the coefficients which is equal to it; but this consists of the algebraic sum of the numerical factors when in it all the coefficients a_0, a_1, a_2, &c. are made equal to unity. In Exs. 13, 15, 16, 18, 20 of Art. 27 we have instances of this theorem.

Examples.

1. Find in terms of p, q, r the value of the symmetric function

$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta},$$

where α, β, γ are the roots of the cubic equation

$$x^3+px^2+qx+r=0.$$

Ans. $\frac{pq}{r}-3.$

2. Find for the same equation the value of

$$(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3.$$

Ans. $24r-p^3.$

3. Calculate the value of $\Sigma\alpha^3\beta^3$ of the roots of the same equation. Here $\Sigma\alpha\beta\Sigma\alpha^2\beta^2=\Sigma\alpha^3\beta^3+\alpha\beta\gamma\Sigma\alpha^2\beta$; hence, &c.

Ans. $q^3-3pqr+3r^2.$

4. Find for the same equation the value of the symmetric function

$$(\beta^3-\gamma^3)^2+(\gamma^3-\alpha^3)^2+(\alpha^3-\beta^3)^2.$$

$\Sigma\alpha^6$ is easily obtained by squaring $\Sigma\alpha^3$ (see Ex. 3, Art. 27)

Ans. $2p^6-12p^4q+12p^3r+18p^2q^2-18pqr-6q^3.$

5. Find for the same equation the value of

$$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}.$$

Ans. $\frac{2p^2q-4pr-2q^2}{r-pq}.$

6. Find for the same equation the value of

$$\frac{\alpha^2+\beta\gamma}{\beta+\gamma}+\frac{\beta^2+\gamma\alpha}{\gamma+\alpha}+\frac{\gamma^2+\alpha\beta}{\alpha+\beta}.$$

Ans. $\frac{p^4-3p^2q+5pr+q^2}{r-pq}$

7. Find for the same equation the value of

$$\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}+\frac{2\gamma\alpha-\beta^2}{\gamma+\alpha-\beta}+\frac{2\alpha\beta-\gamma^2}{\alpha+\beta-\gamma}.$$

Ans. $\frac{p^4-2p^2q+14pr-8q^2}{4pq-p^3-8r}.$

8. Find the value of the symmetric function $\Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$ for the same cubic equation.

$$\textit{Ans.}\quad \frac{3p^2q^2 - 4p^3r - 4q^3 - 2pqr - 9r^2}{(r-pq)^2}.$$

9. Calculate in terms of p, q, r, s the value of $\Sigma\frac{\alpha\beta}{\gamma^2}$ for the equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Here $\quad \Sigma\alpha\beta\Sigma\frac{1}{\alpha^2} = \Sigma\frac{\alpha}{\beta} + \Sigma\frac{\alpha\beta}{\gamma^2}$; and $\Sigma\alpha\Sigma\frac{1}{\alpha} = 4 + \Sigma\frac{\alpha}{\beta}$.

$$\textit{Ans.}\quad \frac{qr^2 - 2q^2s - prs + 4s^2}{s^2}.$$

10. Find the value of $\Sigma\frac{\alpha}{\beta}$ of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0.$$

$$\textit{Ans.}\quad \frac{p_{n-1}p_n - p_1p_{n-1}^2 + 2p_1p_{n-2}p_n}{p_n^2}.$$

11. Find for the biquadratic of Ex. 9 the value of

$$(\beta\gamma - \alpha\delta)(\gamma\alpha - \beta\delta)(\alpha\beta - \gamma\delta).$$

Compare Ex. 22, Art. 24. $\qquad$ *Ans.* $r^2 - p^2s$.

12. Find the value of $\Sigma(a_0\alpha + a_1)^2(\beta-\gamma)^2$ in terms of the coefficients of the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

$$\textit{Ans.}\quad \frac{18}{a_0^2}(a_0a_2 - a_1^2)^2.$$

13. Find the value of the symmetric function $\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$ of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0.$$

The given function may be written in the form

$$\begin{aligned}
&\alpha_1\left\{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ldots + \frac{1}{\alpha_n}\right\} - 1\\
+\ &\alpha_2\left\{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ldots + \frac{1}{\alpha_n}\right\} - 1\\
+\ &\ldots\ldots\ldots\\
+\ &\alpha_n\left\{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ldots + \frac{1}{\alpha_n}\right\} - 1,
\end{aligned}$$

or $\Sigma\alpha_1\Sigma\frac{1}{\alpha_1} - n$; hence, &c. $\qquad$ *Ans.* $\frac{p_1p_{n-1}}{p_n} - n$.

14. Clear of radicals the equation

$$\sqrt{t-\alpha^2} + \sqrt{t-\beta^2} + \sqrt{t-\gamma^2} = 0,$$

and express the coefficients of the resulting equation in t in terms of the coefficients of the cubic of Ex. 1.

Ans. $3t^2 - 2(p^2 - 2q)t - p^4 + 4p^2q - 8pr = 0$.

15. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic of Ex. 9, prove

$$(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1) = (1-q+s)^2 + (p-r)^2.$$

Substitute in turn each of the roots of the equation $x^2 + 1 = 0$ in the identity of Art. 16, and multiply.

16. Prove the following relation between the roots and coefficients of the general equation of the n^{th} degree:—

$$(\alpha_1^2 + 1)(\alpha_2^2 + 1) \ldots . (\alpha_n^2 + 1) = (1 - p_2 + p_4 - \ldots)^2 + (p_1 - p_3 + \ldots)^2.$$

17. Find the numerical value of

$$(\alpha^2 + 2)(\beta^2 + 2)(\gamma^2 + 2)(\delta^2 + 2),$$

where $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$x^4 - 7x^3 + 8x^2 - 5x + 10 = 0.$$

Substitute in turn for x each root of the equation $x^2 + 2 = 0$, and multiply.

Ans. 166.

18. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

prove

$$a_0^3(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)(\alpha+\delta)(\beta+\delta)(\gamma+\delta) = 16\{6a_1a_2a_3 - a_0a_3^2 - a_1^2a_4\}$$

The symmetric function in question is equal to $(\mu+\nu)(\nu+\lambda)(\lambda+\mu)$, or $\Sigma\lambda\Sigma\mu\nu - \lambda\mu\nu$, where λ, μ, ν have the values of Ex. 17, Art. 27.

19. Calculate the value of the symmetric function $\Sigma(\alpha-\beta)^4$ of the roots of the biquadratic equation of Ex. 9.

Ans. $3p^4 - 16p^2q + 20q^2 + 4pr - 16s$.

20. Show that when the biquadratic is written with binomial coefficients, as in Ex. 18, the value of the symmetric function of the preceding example may be expressed in the following form:—

$$a_0^4\Sigma(\alpha-\beta)^4 = 16\{48(a_0a_2 - a_1^2)^2 - a_0^2(a_0a_4 - 4a_1a_3 + 3a_2^2)\}.$$

21. The distances on a right line of two pairs of points from a fixed origin on the line are the roots (α, β) and (α', β') of the two quadratic equations

$$ax^2 + 2bx + c = 0, \quad a'x^2 + 2b'x + c' = 0;$$

prove that when one pair of the points are the harmonic conjugates of the other pair, the following relation exists:—

$$ac' + a'c - 2bb' = 0.$$

22. The distances of three points A, B, C on a right line from a fixed origin O on the line are the roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0;$$

find the condition that one of the points A, B, C should bisect the distance between the other two.

Compare Ex. 15, Art. 27. *Ans.* $a^2 d - 3abc + 2b^3 = 0$.

23. Retaining the notation of the preceding question, find the condition that the four points O, A, B, C should form a harmonic division.

Ans. $ad^2 - 3bcd + 2c^3 = 0$.

This can be derived from the result of Ex. 22 by changing the roots into their reciprocals, or it can be easily calculated independently.

24. If the roots $(\alpha, \beta, \gamma, \delta)$ of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

be so related that $\alpha - \delta$, $\beta - \delta$, $\gamma - \delta$ are in harmonic progression, prove the relation among the coefficients

$$ace + 2bcd - ad^2 - b^2 e - c^3 = 0.$$

Compare Ex. 18, Art. 27.

25. Form the equation whose roots are

$$-\frac{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}{\alpha + \omega\beta + \omega^2\gamma}, \quad -\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\alpha + \omega^2\beta + \omega\gamma},$$

where $\omega^3 = 1$, and α, β, γ are the roots of the cubic

$$ax^3 + 3bx^2 + 3cx + d = 0.$$

Ans. $(ac - b^2)\, x^2 + (ad - bc)\, x + (bd - c^2) = 0$.

Compare Exs. 13 and 14, Art. 27.

26. Express

$$(2\beta\gamma - \gamma\alpha - \alpha\beta)\,(2\gamma\alpha - \alpha\beta - \beta\gamma)\,(2\alpha\beta - \beta\gamma - \gamma\alpha)$$

as the sum of two cubes.

Ans. $(\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta)^3 + (\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta)^3$.

Compare Ex. 5, Art. 26.

27. Express

$$(x + y + z)^3 + (x + \omega y + \omega^2 z)^3 + (x + \omega^2 y + \omega z)^3$$

in terms of $x^3 + y^3 + z^3$ and xyz, where $\omega^3 = 1$.

Ans. $3\,(x^3 + y^3 + z^3) + 18xyz$.

28. If

$$(x^3 + y^3 + z^3 - 3xyz)\,(x'^3 + y'^3 + z'^3 - 3x'y'z') \equiv X^3 + Y^3 + Z^3 - 3XYZ,$$

find X, Y, Z in terms of x, y, z; x', y', z'.

Apply Example 4, Art. 26.

Ans. $X = xx' + yy' + zz'$, $Y = xy' + yz' + zx'$, $Z = xz' + yx' + zy'$.

29. Resolve

$$(\alpha + \beta + \gamma)^3\,\alpha\beta\gamma - (\beta\gamma + \gamma\alpha + \alpha\beta)^3$$

into three factors, each of the second degree in α, β, γ.

Ans. $(\alpha^2 - \beta\gamma)\,(\beta^2 - \gamma\alpha)\,(\gamma^2 - \alpha\beta)$

Compare Ex. 18, Art. 24.

30. Resolve into simple factors each of the following expressions :—

(1). $(\beta-\gamma)^2(\beta+\gamma-2\alpha)+(\gamma-\alpha)^2(\gamma+\alpha-2\beta)+(\alpha-\beta)^2(\alpha+\beta-2\gamma)$.

(2). $(\beta-\gamma)(\beta+\gamma-2\alpha)^2+(\gamma-\alpha)(\gamma+\alpha-2\beta)^2+(\alpha-\beta)(\alpha+\beta-2\gamma)^2$.

Ans. (1). $(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$.
(2). $-9(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$.

31. Find the condition that the cubic equation

$$x^3-px^2+qx-r=0$$

should have a pair of roots of the form $a \pm a\sqrt{-1}$; and show how to determine the roots in that case.

If the real root is b, we easily find, by forming the sum of the squares of the roots, $p^2-2q=b^2$. The required condition is

$$(p^2-2q)(q^2-2pr)-r^2=0.$$

32. Solve the equation

$$x^3-7x^2+20x-24=0,$$

whose roots are of the form indicated in Ex. 31.

Ans. Roots 3, and $2 \pm 2\sqrt{-1}$.

33. Find the conditions that the biquadratic equation

$$x^4-px^3+qx^2-rx+s=0$$

should have roots of the form $a \pm a\sqrt{-1}$, $b \pm b\sqrt{-1}$. Here there must be two conditions among the coefficients, as there are only two independent quantities involved in the roots.

Ans. $p^2-2q=0$; $r^2-2qs=0$.

34. Solve the biquadratic

$$x^4+4x^3+8x^2-120x+900=0,$$

whose roots are of the form in Ex. 33.

Ans. $3 \pm 3\sqrt{-1}$, $-5 \mp 5\sqrt{-1}$.

35. If $\alpha+\beta\sqrt{-1}$ be a root of the equation

$$x^3+qx+r=0,$$

prove that 2α will be a root of the equation

$$x^3+qx-r=0.$$

36. Find the condition that the cubic equation

$$x^3+px^2+qx+r=0$$

should have two roots α, β connected by the relation $\alpha\beta+1=0$.

Ans. $1+q+pr+r^2=0$.

37. Find the condition that the biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0$$

should have two roots connected by the relation $\alpha\beta + 1 = 0$.

The condition arranged according to powers of s is

$$1 + q + pr + r^2 + (p^2 + pr - 2q - 1)\, s + (q - 1)\, s^2 + s^3 = 0.$$

38. Find the value of $\Sigma\, (\alpha_1 - \alpha_2)^2\, \alpha_3\, \alpha_4 \ldots\ldots \alpha_n$ of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0.$$

This is readily reducible to Ex. 13.

Ans. $(-1)^n \{p_1 p_{n-1} - n^2 p_n\}$.

39. If the roots of the equation

$$a_0 x^n + n a_1 x^{n-1} + \frac{n(n-1)}{1\,.\,2}\, a_2 x^{n-2} + \ldots\ldots + a_n = 0$$

be in arithmetical progression, show that they can be obtained from the expression

$$-\frac{a_1}{a_0} \pm \frac{r}{a_0} \sqrt{\frac{3(a_1^2 - a_0 a_2)}{n+1}}$$

by giving to r all the values $1, 3, 5, \ldots\ldots n-1$, when n is even; and all the values $0, 2, 4, 6 \ldots\ldots n-1$, when n is odd.

40. Representing the differences of three quantities α, β, γ by $\alpha_1, \beta_1, \gamma_1$, as follows:—

$$\alpha_1 \equiv \beta - \gamma, \quad \beta_1 \equiv \gamma - \alpha, \quad \gamma_1 \equiv \alpha - \beta;$$

prove the relations

$$\alpha_1^3 + \beta_1^3 + \gamma_1^3 = 3\alpha_1\beta_1\gamma_1,$$

$$\alpha_1^4 + \beta_1^4 + \gamma_1^4 = \tfrac{1}{2}\{\alpha_1^2 + \beta_1^2 + \gamma_1^2\}^2,$$

$$\alpha_1^5 + \beta_1^5 + \gamma_1^5 = \tfrac{5}{2}\{\alpha_1^2 + \beta_1^2 + \gamma_1^2\}\, \alpha_1\beta_1\gamma_1.$$

These results can be derived by taking $\alpha_1, \beta_1, \gamma_1$ to be roots of the equation

$$x^3 + qx - r = 0$$

(where the second term is absent since the sum of the roots $= 0$), and calculating the symmetric functions $\Sigma\alpha_1^3$, $\Sigma\alpha_1^4$, $\Sigma\alpha_1^5$ in terms of q and r. The process can be extended to form $\Sigma\alpha_1^6$, $\Sigma\alpha_1^7$, &c. The sums of the successive powers are, therefore, all capable of being expressed in terms of the product $\alpha_1\beta_1\gamma_1$ and the sum of squares $\alpha_1^2 + \beta_1^2 + \gamma_1^2$; the former being equal to r, and the latter to $-2(\beta_1\gamma_1 + \gamma_1\alpha_1 + \alpha_1\beta_1)$, or $-2q$. These sums can be calculated readily as follows:—By means of $x^3 = r - qx$, and the equations derived from this by squaring, cubing, &c., and multiplying by x or x^2, any power of x, say x^p, can be brought by successive reductions to the form $A + Bx + Cx^2$, where A, B, C are functions of q and r. Substituting $\alpha_1, \beta_1, \gamma_1$, and adding, we find $\Sigma\alpha_1^p = 3A - 2qC$. The student can take as an exercise to prove in this way $\Sigma\alpha_1^7 = 7q^2 r$, $\Sigma\alpha_1^{11} = 11qr(q^3 - r^2)$.

CHAPTER IV.

TRANSFORMATION OF EQUATIONS.

29. **Transformation of Equations.**—We can in many instances, without knowing the values of the roots of an equation in terms of the coefficients, transform it by elementary substitutions, or by the aid of the symmetric functions of the roots, into another equation whose roots shall have certain assigned relations to the roots of the proposed. A transformation of this nature often facilitates the discussion of the equation. We proceed to explain the most important elementary transformations of equations.

30. **Roots with Signs changed.**—To transform an equation into another whose roots shall be equal to the roots of the given equation with contrary signs, let $a_1, a_2, a_3, \ldots a_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots . p_{n-1}x + p_n = 0.$$

We have then the identity

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n \equiv (x - a_1)(x - a_2) \ldots (x - a_n);$$

changing x into $-y$, we have, whether n be even or odd,

$$y^n - p_1y^{n-1} + p_2y^{n-2} - \ldots \pm p_{n-1}y \mp p_n \equiv (y + a_1)(y + a_2) \ldots (y + a_n).$$

The polynomial in y equated to zero is, therefore, an equation whose roots are $-a_1, -a_2, \ldots -a_n$; and to effect the required transformation we have only to *change the signs of every alternate term of the given equation beginning with the second.*

Examples.

1. Find the equation whose roots are the roots of

$$x^5 + 7x^4 + 7x^3 - 8x^2 + x + 1 = 0$$

with their signs changed. *Ans.* $x^5 - 7x^4 + 7x^3 + 8x^2 + x - 1 = 0.$

2. Change the signs of the roots of the equation

$$x^7 + 3x^5 + x^3 - x^2 + 7x + 2 = 0.$$

[Supply the missing terms with zero coefficients.]

Ans. $x^7 + 3x^5 + x^3 + x^2 + 7x - 2 = 0.$

31. **To Multiply the Roots by a Given Quantity.**—To transform an equation whose roots are $a_1, a_2, \ldots a_n$ into another whose roots are $ma_1, ma_2, \ldots ma_n$, we change x into $\frac{y}{m}$ in the identity of the preceding Article. Multiplying by m^n, we have

$$y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + \ldots + m^{n-1}p_{n-1}y + m^np_n$$
$$\equiv (y - ma_1)(y - ma_2) \ldots . (y - ma_n).$$

Hence, to multiply the roots of an equation by a given quantity m, *we have only to multiply the successive coefficients, beginning with the second, by $m, m^2, m^3, \ldots m^n$.*

The present transformation is useful for the purpose of removing the coefficient of the first term of an equation when it it is not unity; and generally for removing fractional coefficients from an equation. If there be a coefficient a_0 of the first term, we form the equation whose roots are $a_0a_1, a_0a_2, \ldots a_0a_n$; the transformed equation will be divisible by a_0, and after such division the coefficient of x^n will be unity.

When there are fractional coefficients, we can get rid of them by multiplying the roots by a quantity m which is the least common multiple of all the denominators of the fractions. In many cases multiplication by a quantity less than the least common multiple will be sufficient for this purpose, as will appear in the following examples:—

Examples.

1. Change the equation

$$3x^4 - 4x^3 + 4x^2 - 2x + 1 = 0$$

into another the coefficient of whose highest term will be unity.

We multiply the roots by 3. *Ans.* $x^4 - 4x^3 + 12x^2 - 18x + 27 = 0.$

2. Remove the fractional coefficients from the equation

$$x^3 - \frac{1}{2}x^2 + \frac{2}{3}x - 1 = 0.$$

Multiply the roots by 6. *Ans.* $x^3 - 3x^2 + 24x - 216 = 0.$

3. Remove the fractional coefficients from the equation

$$x^3 - \frac{5}{2}x^2 - \frac{7}{18}x + \frac{1}{108} = 0.$$

By noting the factors which occur in the denominators of these fractions, we observe that a number much smaller than the least common multiple will suffice to remove the fractions. If the required multiplier be m, we write the transformed equation thus:—

$$x^3 - m\frac{5}{2}x^2 - m^2\frac{7}{3^2 . 2}x + \frac{m^3}{3^3 . 2^2} = 0;$$

it is evident that if m be taken $= 6$, each coefficient will become integral; hence we have only to multiply the roots by 6.

Ans. $x^3 - 15x^2 - 14x + 2 = 0.$

4. Remove the fractional coefficients from the equation

$$x^4 + \frac{3}{10}x^2 + \frac{13}{25}x + \frac{77}{1000} = 0.$$

The student must be careful in examples of this kind to supply the missing terms with zero coefficients. The required multiplier is 10.

Ans. $x^4 + 30x^2 + 520x + 770 = 0.$

5. Remove the fractional coefficients from the equation

$$x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{13}{900} = 0.$$

Ans. $x^4 - 25x^3 + 375x^2 - 11700 = 0.$

32. **Reciprocal Roots and Reciprocal Equations.**— To transform an equation into one whose roots are the reciprocals of the roots of the proposed equation, we change x into $\frac{1}{y}$ in the identity of Art. 30. This substitution gives, after certain easy reductions,

$$\frac{1}{y^n} + \frac{p_1}{y^{n-1}} + \frac{p_2}{y^{n-2}} + \ldots + \frac{p_{n-1}}{y} + p_n \equiv \frac{p_n}{y^n}\left(y - \frac{1}{a_1}\right)\left(y - \frac{1}{a_2}\right)\ldots\left(y - \frac{1}{a_n}\right),$$

or

$$y^n + \frac{p_{n-1}}{p_n}y^{n-1} + \frac{p_{n-2}}{p_n}y^{n-2} + \ldots + \frac{p_1}{p_n}y + \frac{1}{p_n} \equiv \left(y - \frac{1}{a_1}\right)\left(y - \frac{1}{a_2}\right)\ldots\left(y - \frac{1}{a_n}\right);$$

hence, if in the given equation we replace x by $\frac{1}{y}$, and multiply by y^n, the resulting polynomial in y equated to zero will have for roots the reciprocals of $a_1, a_2, \ldots a_n$.

There is a certain class of equations which remain unaltered when x is changed into its reciprocal. These are called *reciprocal equations.* The conditions which must obtain among the coefficients of an equation in order that it should be one of this class are, by what has been just proved, plainly the following:—

$$\frac{p_{n-1}}{p_n} = p_1, \quad \frac{p_{n-2}}{p_n} = p_2, \quad \&c., \quad \frac{p_1}{p_n} = p_{n-1}, \quad \frac{1}{p_n} = p_n.$$

The last of these conditions gives $p_n^2 = 1$, or $p_n = \pm 1$. Reciprocal equations are divided into two classes, according as p_n is equal to $+1$, or to -1.

(1). In the first case we have the relations

$$p_{n-1} = p_1, \; p_{n-2} = p_2, \; . \; . \; . \; p_1 = p_{n-1};$$

which give rise to the *first class of reciprocal equations, in which the coefficients of the corresponding terms taken from the beginning and end are equal in magnitude and have the same signs.*

(2). In the second case, when $p_n = -1$, we have

$$p_{n-1} = -p_1, \; p_{n-2} = -p_2, \; \&c, \; . \; . \; . \; . \; p_1 = -p_{n-1};$$

giving rise to the *second class of reciprocal equations, in which corresponding terms counting from the beginning and end are equal in magnitude but different in sign.* It is to be observed that in this case when the degree of the equation is even, say $n = 2m$, one of the conditions becomes $p_m = -p_m$, or $p_m = 0$; so that in reciprocal equations of the second class, whose degree is even, the middle term is absent.

If a be a root of a reciprocal equation, $\dfrac{1}{a}$ must also be a root, for it is a root of the transformed equation, and the transformed equation is identical with the proposed; hence the roots of a reciprocal equation occur in pairs, $a, \dfrac{1}{a}$; $\beta, \dfrac{1}{\beta}$; &c. When the degree is odd there must be a root which is its own reciprocal; and it is in fact obvious from the form of the equation that -1, or $+1$ is then a root, according as the equation is of the first or second of the above classes. In either case we can divide off by

the known factor ($x+1$ or $x-1$), and what is left is a reciprocal equation of even degree and of the first class. In equations of the second class of even degree x^2-1 is a factor, since the equation may be written in the form

$$x^n - 1 + p_1 x(x^{n-2}-1) + \ldots = 0.$$

By dividing by x^2-1, this also is reducible to a reciprocal equation of the first class of even degree. Hence all reciprocal equations may be reduced to *those of the first class whose degree is even*, and this may consequently be regarded as *the standard form of reciprocal equations.*

EXAMPLES.

1. Find the equation whose roots are the reciprocals of the roots of

$$x^4 - 3x^3 + 7x^2 + 5x - 2 = 0.$$

Ans. $2y^4 - 5y^3 - 7y^2 + 3y - 1 = 0.$

2. Reduce to a reciprocal equation of even degree and of first class

$$x^6 + \frac{5}{6}x^5 - \frac{22}{3}x^4 + \frac{22}{3}x^2 - \frac{5}{6}x - 1 = 0.$$

Ans. $x^4 + \frac{5}{6}x^3 - \frac{19}{3}x^2 + \frac{5}{6}x + 1 = 0.$

33. **To Increase or Diminish the Roots by a Given Quantity.**—To effect this transformation we change the variable in the polynomial $f(x)$ by the substitution $x = y + h$; the resulting equation in y will have roots each less or greater by h than the given equation in x, according as h is positive or negative. The resulting equation is (see Art. 6)

$$f(h) + f'(h)\,y + \frac{f''(h)}{1.2}y^2 + \frac{f'''(h)}{1.2.3}y^3 + \ldots = 0.$$

There is a mode of formation of this equation which for practical purposes is much more convenient than the direct calculation of the derived functions, and the substitution in them of the given quantity h. This we proceed to explain. Let the proposed equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0;$$

and suppose the transformed polynomial in y to be

$$A_0 y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_{n-1} y + A_n ;$$

since $y = x - h$, this is equivalent to

$$A_0 (x-h)^n + A_1 (x-h)^{n-1} + \ldots + A_{n-1} (x-h) + A_n,$$

which must be identical with the given polynomial. We conclude that if the given polynomial be divided by $x - h$, the remainder is A_n, and the quotient

$$A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + \ldots + A_{n-2} (x-h) + A_{n-1} ;$$

if this again be divided by $x - h$, the remainder is A_{n-1}, and the quotient

$$A_0 (x-h)^{n-2} + A_1 (x-h)^{n-3} + \ldots + A_{n-2}.$$

Proceeding in this way, we are able by a repetition of arithmetical operations, of the kind explained in Art. 8, to calculate in succession the several coefficients A_n, A_{n-1}, &c., of the transformed equation; the last, A_0, being equal to a_0. It will appear in a subsequent chapter that the best practical method of solving numerical equations is only an extension of the process employed in the following examples.

Examples.

1. Find the equation whose roots are the roots of

$$x^4 - 5x^3 + 7x^2 - 17x + 11 = 0,$$

each diminished by 4.

The calculation is best exhibited as follows :—

1	− 5	7	− 17	11
	4	− 4	12	− 20
	− 1	3	− 5	− 9
	4	12	60	
	3	15	55	
	4	28		
	7	43		
	4			
	11			

Here the first division of the given polynomial by $x-4$ gives the remainder -9 $(=A_4)$, and the quotient x^3-x^2+3x-5 (cf. Art. 8). Dividing this again by $x-4$, we get the remainder 55 $(=A_3)$, and the quotient $x^2+3x+15$. Dividing again, we get the remainder 43 $(=A_2)$, and quotient $x+7$; and dividing this we get $A_1=11$, and $A_0=1$; hence the required transformed equation is

$$y^4+11y^3+43y^2+55y-9=0.$$

2. Find the equation whose roots are the roots of

$$x^5+4x^3-x^2+11=0,$$

each diminished by 3.

1	0	4	− 1	0	11
	3	9	39	114	342
	3	13	38	114	**353**
	3	18	93	393	
	6	31	131	**507**	
	3	27	174		
	9	58	**305**		
	3	36			
	12	**94**			
	3				
	15				

The transformed equation is, therefore,

$$y^5+15y^4+94y^3+305y^2+507y+353=0.$$

3. Find the equation whose roots are the roots of

$$4x^5-2x^3+7x-3=0,$$

each increased by 2.

The multiplier in this operation is, of course, -2.

Ans. $4y^5-40y^4+158y^3-308y^2+303y-129=0.$

4. Increase by 7 the roots of the equation

$$3x^4+7x^3-15x^2+x-2=0.$$

Ans. $3y^4-77y^3+720y^2-2876y+4058=0.$

5. Diminish by 23 the roots of the equation

$$5x^3-13x^2-12x+7=0.$$

The operation may be conveniently performed by first diminishing the roots by 20, and then diminishing the roots of the transformed equation again by 3. The

calculation may be exhibited in two stages, as follows, the broken lines marking the conclusion of each stage :—

5	− 13	− 12	7
	100	1740	34560
	87	1728	**34567**
	100	3740	19122
	187	**5468**	**53689**
	100	906	
	287	6374	
	15	951	
	302	**7325**	
	15		
	317		
	15		
	332		

Ans. $5y^3 + 332y^2 + 7325y + 53689 = 0.$

34. **Removal of Terms.**—One of the chief uses of the transformation of the preceding article is to remove a certain specified term from an equation. Such a step often facilitates its solution. Writing the transformed equation in descending powers of y, we have

$$a_0y^n + (na_0h + a_1)y^{n-1} + \left\{\frac{n(n-1)}{1.2}a_0h^2 + (n-1)a_1h + a_2\right\}y^{n-2} + \ldots = 0.$$

If h be such as to satisfy the equation $na_0h + a_1 = 0$, the transformed equation will want the second term. If h be either of the values which satisfy the equation

$$\frac{n(n-1)}{1.2}a_0h^2 + (n-1)a_1h + a_2 = 0,$$

the transformed equation will want the third term; the removal of the fourth term will require the solution of a cubic for h; and so on. To remove the last term we must solve the equation $f(h) = 0$, which is the original equation itself.

Examples.

1. Transform the equation

$$x^3 - 6x^2 + 4x - 7 = 0$$

into one which shall want the second term.

$$na_0h + a_1 = 0 \quad \text{gives } h = 2.$$

Diminish the roots by 2. *Ans.* $y^3 - 8y - 15 = 0.$

2. Transform the equation

$$x^4 + 8x^3 + x - 5 = 0$$

into one which shall want the second term.

Increase the roots by 2. *Ans.* $y^4 - 24y^2 + 65y - 55 = 0.$

3. Transform the equation

$$x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$$

into one which shall want the third term.

The quadratic for h is

$$6h^2 - 12h - 18 = 0, \text{ giving } h = 3, \ h = -1.$$

Thus there are two ways of effecting the transformation.

Diminishing the roots by 3, we obtain

$$(1) \quad y^4 + 8y^3 - 111y - 196 = 0.$$

Increasing the roots by 1, we obtain

$$(2) \quad y^4 - 8y^3 + 17y - 8 = 0.$$

35. **Binomial Coefficients.**—In many algebraical processes it is found convenient to write the polynomial $f(x)$ in the following form :—

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + \frac{n(n-1)}{1.2}a_{n-2}x^2 + na_{n-1}x + a_n,$$

in which each term is affected, in addition to the literal coefficient, with the numerical coefficient of the corresponding term in the expansion of $(x+1)^n$ by the binomial theorem. The student will find examples of equations written in this way on referring to Article 27, Examples 13 and 16. The form is one to which any given polynomial can be at once reduced.

We now adopt the following notation :—

$$U_n \equiv a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + na_{n-1}x + a_n,$$

thus using U with the suffix n to represent the polynomial of the n^{th} degree written with binomial coefficients

We have, therefore, changing n into $n-1$, &c.,

$$U_{n-1} = a_0x^{n-1} + (n-1)\,a_1x^{n-2} + \ldots + (n-1)\,a_{n-2}x + a_{n-1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$U_3 \;\;\equiv a_0x^3 + 3a_1x^2 + 3a_2x + a_3,$$

$$U_2 \;\;\equiv a_0x^2 + 2a_1x + a_2,$$

$$U_1 \;\;\equiv a_0x + a_1,$$

$$U_0 \;\;\equiv a_0.$$

One advantage of the binomial form is, that the derived functions can be immediately written down. The first derived function of U_n is, plainly,

$$n\left\{a_0x^{n-1} + (n-1)\,a_1x^{n-2} + \frac{(n-1)\,(n-2)}{1\,.\,2}\,a_2x^{n-3} + \ldots + a_{n-1}\right\};$$

or nU_{n-1}; so that the first derived function of a polynomial represented in this way can be formed by applying to the suffix of U the rule given in Art. 6 with respect to the exponent of the variable. Thus, for example, the first derived of U_4 is formed by multiplying the function by 4, and diminishing the suffix by unity; it is, therefore, $4U_3$, as the student can easily verify.

We proceed now to prove that the substitution of $y+h$ for x transforms the polynomial U_n, or

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1\,.\,2}\,a_2x^{n-2} + \ldots + na_{n-1}x + a_n,$$

into

$$A_0y^n + nA_1y^{n-1} + \frac{n(n-1)}{1\,.\,2}\,A_2y^{n-2} + \ldots + nA_{n-1}y + A_n,$$

where

$$A_0,\; A_1,\; A_2,\; \ldots\; A_{n-1},\; A_n$$

are the functions which result by substituting h for x in

$$U_0,\; U_1,\; U_2,\; \ldots\; U_{n-1},\; U_n;$$

i.e. $\quad A_0 = a_0, \quad A_1 = a_0h + a_1, \quad A_2 = a_0h^2 + 2a_1h + a_2$, &c.

Representing the derived functions of $f(h)$ by suffixes, as

explained in Art. 6, we may write the result of the transformation, viz. $f(y+h)$, in the following form:—

$$f(h)+f_1(h)y+\frac{f_2(h)}{1.2}y^2+\ldots+\frac{f_{n-1}(h)}{1.2\ldots n-1}y^{n-1}+\frac{f_n(h)}{1.2\ldots n}y^n;$$

$f(h)$ is the result of substituting h for x in U_n; it is, therefore, A_n; its first derived $f_1(h)$ is, by the above rule, nA_{n-1}; the first derived of this again is $n(n-1)A_{n-2}$; and so on. Making these substitutions, we have the result above stated, which enables us to write down without any calculation the transformed equation.

Examples.

1. Find the result of substituting $y+h$ for x in the polynomial

$$a_0x^3+3a_1x^2+3a_2x+a_3.$$

Ans. $a_0y^3+3(a_0h+a_1)y^2+3(a_0h^2+2a_1h+a_2)y+a_0h^3+3a_1h^2+3a_2h+a_3$.

The student will find it a useful exercise to verify this result by the method of calculation explained in Art. 33, which may often be employed with advantage ir the case of algebraical as well as numerical examples.

2. Remove the second term from the equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0.$$

We must diminish the roots by a quantity h obtained from the equation

$$a_0h+a_1=0,\quad \text{i.e. } h=\frac{-a_1}{a_0}.$$

Substituting this value of h in A_2, and A_3, the resulting equation in y is

$$y^3+\frac{3(a_0a_2-a_1^2)}{a_0^2}y+\frac{a_0^2a_3-3a_0a_1a_2+2a_1^3}{a_0^3}=0.$$

3. Find the condition that the second and third terms of the equation $U_n=0$ should be capable of being removed by the same substitution.

Here A_1 and A_2 must vanish for the same value of h; and eliminating h between them we find the required condition.

Ans. $a_0a_2-a_1^2=0$.

4. Solve the equation

$$x^3+6x^2+12x-19=0$$

by removing its second term.

The third term is removed by the same substitution, which gives

$$y^3-27=0.$$

The required roots are obtained by subtracting 2 from each root of the latter equation.

5. Find the condition that the second and fourth terms of the equation $U_n = 0$ should be capable of being removed by the same transformation.

Here the coefficients A_1 and A_3 must vanish for the same value of h; eliminating h between the equations

$$a_0 h + a_1 = 0, \quad a_0 h^3 + 3a_1 h^2 + 3a_2 h + a_3 = 0,$$

we obtain the required condition

$$a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = 0.$$

N.B.—When this condition holds among the coefficients of a biquadratic equation its solution is reducible to that of a quadratic; for when the second term is removed the resulting equation is a quadratic for y^2; and from the values of y those of x can be obtained.

6. Solve the equation

$$x^4 + 16x^3 + 72x^2 + 64x - 129 = 0$$

by removing its second term.

The equation in y is

$$y^4 - 24y^2 - 1 = 0.$$

7. Solve in the same manner the equation

$$x^4 + 20x^3 + 143x^2 + 430x + 462 = 0.$$

Ans. The roots are $-7, \ -3, \ -5 \pm \sqrt{3}$.

8. Find the condition that the same transformation should remove the second and fifth terms of the equation $U_n = 0$.

Ans. $a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4 = 0.$

36. **The Cubic.**—On account of their peculiar interest, we shall consider in this and the next following Articles the equations of the third and fourth degrees, in connexion with the transformation of the preceding article. When $y + h$ is substituted for x in the equation

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0, \qquad (1)$$

we obtain

$$a_0 y^3 + 3A_1 y^2 + 3A_2 y + A_3 = 0,$$

where A_1, A_2, A_3 have the values of Art. 35.

If in the transformed equation the second term be absent,

$$A_1 = 0, \quad \text{or} \quad h = -\frac{a_1}{a_0}.$$

Substituting this value for h in A_2 and A_3, we find, as in Ex. 2, Art. 35,

$$a_0 A_2 = a_0 a_2 - a_1^2, \quad a_0^2 A_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3;$$

hence the transformed cubic, wanting the second term, is

$$y^3 + \frac{3}{a_0^2}(a_0 a_2 - a_1^2)\, y + \frac{1}{a_0^3}(a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3) = 0.$$

The functions of the coefficients here involved are of such importance in the theory of algebraic equations, that it is customary to represent them by single letters. We accordingly adopt the notation

$$a_0 a_2 - a_1^2 \equiv H, \quad a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \equiv G;$$

and write the transformed equation in the form

$$y^3 + \frac{3H}{a_0^2}\, y + \frac{G}{a_0^3} = 0. \tag{2}$$

If the roots of this equation be multiplied by a_0 it becomes

$$z^3 + 3Hz + G = 0: \tag{3}$$

a form which will be found convenient in the subsequent discussion of the cubic. The variable, z, which occurs in the first member of this equation, is equal to $a_0 y$ or $a_0 x + a_1$; the original cubic multiplied by a_0^2 being in fact identical with

$$(a_0 x + a_1)^3 + 3H(a_0 x + a_1) + G,$$

as the student can easily verify.

If the roots of the original equation be α, β, γ, those of the transformed equation (2) will be

$$\alpha + \frac{a_1}{a_0}, \quad \beta + \frac{a_1}{a_0}, \quad \gamma + \frac{a_1}{a_0};$$

or, since

$$\alpha + \beta + \gamma = -\frac{3a_1}{a_0},$$

they may be written as follows:—

$$\tfrac{1}{3}(2\alpha - \beta - \gamma), \quad \tfrac{1}{3}(2\beta - \gamma - \alpha), \quad \tfrac{1}{3}(2\gamma - \alpha - \beta).$$

We can write down immediately by the aid of the transformed equation the values of the symmetric functions

$$\Sigma (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha),\ (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta)$$

of the roots of the original cubic. The latter will be found to agree with the value already found in Ex. 15, Art. 27.

We may here make with regard to the general equation an important observation: that any symmetric function of the roots $\alpha, \beta, \gamma, \delta$, &c., which is a function of their *differences* only, can be expressed by the functions of the coefficients which occur in the transformed equation wanting the second term. This is obvious, since the difference of any two roots α', β' of the transformed equation is equal to the difference of the two corresponding roots α, β of the original equation; and any symmetric function of the roots $\alpha', \beta', \gamma', \delta'$, &c., can be expressed in terms of the coefficients of the transformed equation. For example, in the case of the cubic, all symmetric functions of the roots which contain the differences only can be expressed as functions of a_0, H, and G. Illustrations of this principle will be found among the examples of Art. 27.

37. **The Biquadratic.**—The transformed equation, wanting the second term, is in this case

$$a_0 y^4 + 6A_2 y^2 + 4A_3 y + A_4 = 0,$$

where A_2 and A_3 have the same values as in the preceding article; and where A_4 is given by the equation

$$a_0^3 A_4 = a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4.$$

The transformed equation is, therefore,

$$y^4 + \frac{6}{a_0^2} Hy^2 + \frac{4}{a_0^3} Gy + \frac{1}{a_0^4}(a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4) = 0.$$

We might if we pleased represent the absolute term of this equation by a symbol like H and G, and have thus three functions of the coefficients, in terms of which all symmetric functions of the differences of the roots of the biquadratic could be expressed. It is more convenient, however, to regard this

term as composed of H and another function of the coefficients determined in the following manner:—We have plainly the identity

$$a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4 = a_0^2(a_0a_4 - 4a_1a_3 + 3a_2^2) - 3\,(a_0a_2 - a_1^2)^2.$$

This involves a_0, H, and another function of the coefficients, viz.

$$a_0a_4 - 4a_1a_3 + 3a_2^2,$$

which is of great importance in the theory of the biquadratic. This function is represented by the letter I, giving

$$a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4 = a_0^2I - 3H^2.$$

The transformed equation may now be written

$$y^4 + \frac{6H}{a_0^2}y^2 + \frac{4G}{a_0^3}y + \frac{a_0^2I - 3H^2}{a_0^4} = 0. \qquad (1)$$

We can multiply the roots of this equation, as in the case of the cubic of Art. 36, by a_0; and obtain

$$z^4 + 6Hz^2 + 4Gz + a_0^2I - 3H^2 = 0. \qquad (2)$$

This form will be found convenient in the treatment of the algebraical solution of the biquadratic. The variable is the same as in the case of the cubic, viz. $a_0x + a_1$; the original quartic multiplied by a_0^3 being in fact identical with

$$(a_0x + a_1)^4 + 6H(a_0x + a_1)^2 + 4G\,(a_0x + a_1) + a_0^2I - 3H^2.$$

Any symmetric function of the roots of the original biquadratic equation which contains their differences only can therefore be expressed by a_0, H, G, and I.

If the roots of the original equation be α, β, γ, δ, those of the transformed (1) will be, as is easily seen,

$$\tfrac{1}{4}(3\alpha-\beta-\gamma-\delta),\ \tfrac{1}{4}(3\beta-\gamma-\delta-\alpha),\ \tfrac{1}{4}(3\gamma-\delta-\alpha-\beta),\ \tfrac{1}{4}(3\delta-\alpha-\beta-\gamma).$$

The sum of these $= 0$; the sum of their products in pairs $= \dfrac{6H}{a_0^2}$; the sum of their products in threes $= \dfrac{-4G}{a_0^3}$; and for their

continued product we have the equation

$$a_0^4(3\alpha - \beta - \gamma - \delta)(3\beta - \gamma - \delta - \alpha)(3\gamma - \delta - \alpha - \beta)(3\delta - \alpha - \beta - \gamma)$$
$$= 256(a_0^2 I - 3H^2).$$

There is another function of the coefficients to which we wish now to call attention, as it will be found to be of great importance in the subsequent discussion of the biquadratic. It is the function arrived at in Ex. 18, Art. 27, viz.

$$a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.$$

This is denoted by the letter J. The example referred to shows that it is a function of the differences of the roots. It must, therefore, be capable of being expressed in terms of a_0, H, G, and I. We have, in fact, the identity

$$a_0^3 J \equiv a_0^2 HI - G^2 - 4H^3,$$

which the student can easily verify.

Or this relation can be derived as follows:—Whenever a function of the coefficients a_0, a_1, a_2, &c., is the expression of a function of the differences of the roots, it must be unaltered by the transformation which removes the second term of the equation; hence its value is unaltered when we change a_1 into zero, a_2 into A_2, a_3 into A_3, &c. Thus

$$a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3 \equiv a_0A_2A_4 - a_0A_3^2 - A_2^3;$$

substituting for A_2, A_3, A_4 their values in terms of H, G, I, we easily obtain the above identity, which will usually be written in the form

$$G^2 + 4H^3 \equiv a_0^2(HI - a_0J).$$

38. **Homographic Transformation.**—The transformation of a polynomial considered in Art. 33 is a particular case of the following, in which x is connected with the new variable y by the equation

$$y = \frac{\lambda x + \mu}{\lambda' x + \mu'}.$$

If $\lambda = 1$, $\mu = -h$, $\lambda' = 0$, $\mu' = 1$, we have $y = x - h$, as in Art. 33. Solving for x in terms of y, we have

$$x = \frac{\mu - \mu' y}{\lambda' y - \lambda}.$$

This value can be substituted for x in the given equation, and the resulting equation of the n^{th} degree in y obtained.

Let $\alpha, \beta, \gamma, \delta$, &c., be the roots of the original equation, and $\alpha', \beta', \gamma', \delta'$, &c., the corresponding roots of the transformed equation. From the equations

$$\alpha' = \frac{\lambda\alpha + \mu}{\lambda'\alpha + \mu'}, \quad \beta' = \frac{\lambda\beta + \mu}{\lambda'\beta + \mu'}, \text{ \&c.,}$$

we easily derive the relation

$$\alpha' - \beta' = \frac{(\lambda\mu' - \lambda'\mu)(\alpha - \beta)}{(\lambda'\alpha + \mu')(\lambda'\beta + \mu')};$$

with corresponding relations for the differences of any other pair of roots. If we take any four roots, and the four corresponding roots, we obtain the equation

$$\frac{(\alpha' - \beta')(\gamma' - \delta')}{(\alpha' - \gamma')(\beta' - \delta')} = \frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \gamma)(\beta - \delta)}.$$

Thus, if the roots of the proposed equation represent the distances of a number of points on a right line from a fixed origin on the line, the roots of the transformed equation will represent the distances of a corresponding system of points, so related to the former that the anharmonic ratio of any four of one system is the same as that of their four conjugates in the other system. It is in consequence of this property that the transformation is called *homographic.*

It is important to observe that the transformation here considered, in which the variables x and y are connected by a relation of the form

$$Axy + Bx + Cy + D = 0,$$

is the most general transformation in which to one value of either variable corresponds one, and only one, value of the other.

39. **Transformation by Symmetric Functions.**—Suppose it is required to transform an equation into another whose roots shall be given rational functions of the roots of the proposed. Let the given function be $\phi\ (\alpha, \beta, \gamma \ldots)$, where ϕ may involve all the roots, or any number of them. We form all pos-

sible combinations $\phi(\alpha\beta\gamma)$, $\phi(\alpha\beta\delta)$, &c., of the roots of this type, and write down the transformed equation as follows :—

$$\{y - \phi(\alpha\beta\gamma \ldots)\}\{y - \phi(\alpha\beta\delta \ldots)\} \ldots = 0.$$

When this product is expanded, the successive coefficients of y will be symmetric functions of the roots α, β, γ, &c., of the given equation ; and may therefore be expressed in terms of the coefficients of that equation.

Examples.

1. The roots of

$$x^3 + px^2 + qx + r = 0$$

are α, β, γ ; find the equation whose roots are α^2, β^2, γ^2.

Suppose the transformed equation to be

$$y^3 + Py^2 + Qy + R = 0 ;$$

then

$$-P = \alpha^2 + \beta^2 + \gamma^2, \quad Q = \Sigma\alpha^2\beta^2, \quad -R = \alpha^2\beta^2\gamma^2 ;$$

and we have to form the symmetric functions $\Sigma\alpha^2$, $\Sigma\alpha^2\beta^2$, $\alpha^2\beta^2\gamma^2$, of the given equation. We easily obtain

$$\Sigma\alpha^2 = p^2 - 2q, \quad \Sigma\alpha^2\beta^2 = q^2 - 2pr, \quad \alpha^2\beta^2\gamma^2 = r^2 ;$$

the transformed equation is, therefore,

$$y^3 - (p^2 - 2q)y^2 + (q^2 - 2pr)y - r^2 = 0.$$

2. Find in the same case the equation whose roots are α^3, β^3, γ^3.

Ans. $y^3 + (p^3 - 3pq + 3r)y^2 + (q^3 - 3pqr + 3r^2)y + r^3 = 0.$

3. If α, β, γ, δ be the roots of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

find the equation whose roots are α^2, β^2, γ^2, δ^2.

Let the transformed equation be

$$y^4 + Py^3 + Qy^2 + Ry + S = 0,$$

then

$$-P = \Sigma\alpha^2, \quad Q = \Sigma\alpha^2\beta^2, \quad -R = \Sigma\alpha^2\beta^2\gamma^2, \quad S = \alpha^2\beta^2\gamma^2\delta^2.$$

Compare Exs. 8, 17, Art. 27.

Ans. $y^4 - (p^2 - 2q)y^3 + (q^2 - 2pr + 2s)y^2 - (r^2 - 2qs)y + s^2 = 0.$

4. If α, β, γ, δ be the roots of

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 ;$$

find the equation whose roots are λ, μ, ν ; viz.,

$$\beta\gamma + \alpha\delta, \quad \gamma\alpha + \beta\delta, \quad \alpha\beta + \gamma\delta.$$

See Ex. 17, Art. 27.

Ans. $y^3 - \dfrac{6a_2}{a_0}y^2 + \dfrac{4}{a_0^2}(4a_1a_3 - a_0a_4)y - \dfrac{8}{a_0^3}(2a_0a_3^2 - 3a_0a_2a_4 + 2a_1^2a_4) = 0.$

5. Show that the transformed equation, when the roots of the resulting cubic of Ex. 4 are multiplied by $\frac{1}{2}a_0$, and the second term of the equation then removed, is

$$z^3 - Iz + 2J = 0.$$

40. **Formation of the Equation whose Roots are any Powers of the Roots of the Proposed.**—The method of effecting this transformation by symmetric functions, as explained in the preceding article, is often laborious. A much simpler process, involving multiplication only, can be employed. It depends on a knowledge of the solution of the binomial equation $x^n - 1 = 0$. This form of equation will be discussed in the next chapter. The general process will be sufficiently obvious to the student from the application to the equations of the 2nd and 3rd degrees which will be found among the following examples:—

EXAMPLES.

1. Form the equation whose roots are the squares of the roots of

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0.$$

To effect this transformation, we have the identity

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n \equiv (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n);$$

changing x into $-x$, we derive, as in Art. 30,

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots \pm p_{n-1} x \mp p_n \equiv (x + \alpha_1)(x + \alpha_2) \ldots (x + \alpha_n);$$

multiplying, we have

$$(x^n + p_2 x^{n-2} + p_4 x^{n-4} + \ldots)^2 - (p_1 x^{n-1} + p_3 x^{n-3} + \ldots)^2 \equiv (x^2 - \alpha_1^2)(x^2 - \alpha_2^2) \ldots (x^2 - \alpha_n^2);$$

it is evident that the first member of this identity contains, when expanded, only even powers of x; we may then replace x^2 by y, and obtain finally

$$y^n + (2p_2 - p_1^2) y^{n-1} + (p_2^2 - 2p_1 p_3 + 2p_4) y^{n-2} + \ldots \equiv (y - \alpha_1^2)(y - \alpha_2^2) \ldots (y - \alpha_n^2).$$

The first member of this equated to zero is the required transformed equation.

N.B.—This transformation will often enable us to determine a limit to the number of real roots of the proposed equation. For, the square of a real root must be positive; and therefore the original equation cannot have more real roots than the transformed has positive roots.

2. Find the equation whose roots are the squares of the roots of

$$x^3 - x^2 + 8x - 6 = 0.$$

Ans. $y^3 + 15y^2 + 52y - 36 = 0.$

The latter equation, by Descartes' rule of signs, cannot have more than one positive root; hence the former must have a pair of imaginary roots.

3. Find the equation whose roots are the squares of the roots of

$$x^5 + x^3 + x^2 + 2x + 3 = 0.$$

Ans. $y^5 + 2y^4 + 5y^3 + 3y^2 - 2y - 9 = 0.$

It follows from Descartes' rule of signs that the original equation must have four imaginary roots.

4. Verify by the method of Ex. 1 the Examples 1 and 3 of Art. 39.

5. Form the equation whose roots are the cubes of the roots of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

It will be observed that in Ex. 1 the process consists in multiplying together $f(x)$, the given polynomial, and $f(-x)$: the variables involved in these being those which are obtained by multiplying x by the two roots of the equation $x^2 - 1 = 0$. In the present case we must multiply together $f(x)$, $f(\omega x)$, $f(\omega^2 x)$: the variables involved being obtained by multiplying x by the roots of the equation $x^3 - 1 = 0$. The transformation may be conveniently represented as follows:—

Write the polynomial $f(x)$ in the form

$$(p_n + p_{n-3}x^3 + \ldots) + x(p_{n-1} + p_{n-4}x^3 + \ldots) + x^2(p_{n-2} + p_{n-5}x^3 + \ldots),$$

which we represent, for brevity, by

$$P + xQ + x^2R,$$

where P, Q, and R are all functions of x^3.

We have then

$$P + xQ + x^2R \equiv (x - \alpha_1)(x - \alpha_2) \ldots . (x - \alpha_n). \qquad (1)$$

Changing, in this identity, x into ωx and $\omega^2 x$ successively, we obtain

$$P + \omega xQ + \omega^2x^2R \equiv (\omega x - \alpha_1)(\omega x - \alpha_2) \ldots (\omega x - \alpha_n), \qquad (2)$$

$$P + \omega^2xQ + \omega x^2R \equiv (\omega^2x - \alpha_1)(\omega^2x - \alpha_2) \ldots (\omega^2x - \alpha_n), \qquad (3)$$

since P, Q, and R, being functions of x^3, are unaltered.

Multiplying together both members of (1), (2), (3), and attending to the results of Art. 26, we obtain

$$P^3 + x^3Q^3 + x^6R^3 - 3x^3PQR \equiv (x^3 - \alpha_1^3)(x^3 - \alpha_2^3) \ldots (x^3 - \alpha_n^3).$$

The first member of this identity contains x in powers which are multiples of 3 only. We can, therefore, substitute y for x^3 and obtain the required transformed equation.

6. Find the equation whose roots are the cubes of the roots of

$$x^4 - x^3 + 2x^2 + 3x + 1 = 0.$$

Ans. $y^4 + 14y^3 + 50y^2 + 6y + 1 = 0.$

7. Verify by the method of Ex. 5 the result of Ex. 2 of Art. 39.

8. Form the equation whose roots are the cubes of the roots of

$$ax^3 + 3bx^2 + 3cx + d = 0.$$

Ans. $a^2y^3 + 3(a^2d + 9b^3 - 9abc)y^2 + 3(ad^2 + 9c^3 - 9bcd)y + d^3 = 0.$

41. **Transformation in General.**—In the general problem of transformation we have to form a new equation in y, whose roots are connected by a given relation $\phi(x, y) = 0$ with the roots of the proposed equation $f(x) = 0$. The transformed equation will then be obtained by substituting in the given equation the value of x in terms of y derived from the given relation $\phi(x, y) = 0$; or, in other words, by eliminating x between the two equations $f(x) = 0$, and $\phi(x, y) = 0$. For example, suppose it were required to form the equation whose roots are the sums of every two of the roots (α, β, γ) of the cubic

$$x^3 - px^2 + qx - r = 0.$$

We have here

$$y = \beta + \gamma = \alpha + \beta + \gamma - \alpha = p - \alpha.$$

The equation $\phi(x, y) = 0$ is in this case $y = p - x$; for when x takes the value α, y takes one of the proposed values; and when x takes the values β and γ, y takes the other proposed values. The transformed equation is therefore obtained by substituting $p - y$ for x in the given equation.

Examples.

1. If α, β, γ be the roots of the cubic

$$x^3 - px^2 + qx - r = 0,$$

form the equation whose roots are

$$\beta\gamma + \frac{1}{\alpha}, \quad \gamma\alpha + \frac{1}{\beta}, \quad \alpha\beta + \frac{1}{\gamma}.$$

Here

$$y = \beta\gamma + \frac{1}{\alpha} = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{1 + r}{\alpha};$$

and the given relation is $xy = 1 + r$; the transformed equation is then obtained by substituting $\frac{1 + r}{y}$ for x in $f(x) = 0$.

Ans. $ry^3 - q(1 + r)y^2 + p(1 + r)^2 y - (1 + r)^3 = 0.$

2. Form, for the same cubic, the equation whose roots are

$$\alpha\beta + \alpha\gamma, \quad \alpha\beta + \beta\gamma, \quad \beta\gamma + \alpha\gamma.$$

Substitute $\frac{r}{q - y}$ for x. *Ans.* $y^3 - 2qy^2 + (pr + q^2)y + r^2 - pqr = 0.$

. Form, for the same cubic, the equation whose roots are

$$\frac{\alpha}{\beta+\gamma-\alpha}, \quad \frac{\beta}{\gamma+\alpha-\beta}, \quad \frac{\gamma}{\alpha+\beta-\gamma}.$$

Substitute $\dfrac{py}{1+2y}$ for x.

Ans. $(p^3-4pq+8r)y^3+(p^3-4pq+12r)y^2+(6r-pq)y+r=0.$

4. If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0,$$

prove that the equation in y whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}, \quad \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}, \quad \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

is obtained by the homographic transformation

$$axy+b(x+y)+c=0.$$

42. **Equation of Squared Differences of a Cubic.**—We shall now apply the transformation explained in the preceding article to an important problem, viz. the formation of the equation whose roots are the squares of the differences of every two of the roots of a given cubic. We shall do this in the first instance for the cubic

$$x^3+qx+r=0, \tag{1}$$

in which the second term is absent, and to which the general equation is readily reducible. Let the roots be α, β, γ. We have to form the equation in y whose roots are

$$(\beta-\gamma)^2, \ (\gamma-\alpha)^2, \ (\alpha-\beta)^2.$$

We may here observe that the method of Art. 39 can be applied to the solution of the general problem, viz. the formation of the equation whose roots are the squares of the differences of every two of the roots of a given equation; for when the product

$$\{y-(\alpha_1-\alpha_2)^2\}\{y-(\alpha_1-\alpha_3)^2\}\{y-(\alpha_1-\alpha_4)^2\} \ldots \ldots \{y-(\alpha_2-\alpha_3)^2\} \ldots$$

is formed, the coefficients of the successive powers of y will be symmetric functions of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, &c., and may, therefore, be expressed in terms of the coefficients of the given equation. In

the present instance, however, the method of Art. 41 leads more readily to the required transformed equation. This equation may be called for brevity the "equation of squared differences" of the proposed equation. Assuming y equal to any one of the roots of the transformed equation, e.g. $(\beta - \gamma)^2$, we have

$$y = (\beta - \gamma)^2 = a^2 + \beta^2 + \gamma^2 - a^2 - \frac{2a\beta\gamma}{a};$$

also

$$a^2 + \beta^2 + \gamma^2 = -2q, \quad a\beta\gamma = -r.$$

The equation $\phi(x, y) = 0$ of Art. 41 becomes, therefore,

$$y = -2q - x^2 + \frac{2r}{x},$$

or

$$x^3 + (y + 2q)x - 2r = 0;$$

subtracting from this the proposed equation, we get

$$(y + q)x - 3r = 0, \quad \text{or} \quad x = \frac{3r}{y + q};$$

hence the transformed equation in y is

$$y^3 + 6qy^2 + 9q^2y + 4q^3 + 27r^2 = 0. \tag{2}$$

If it be proposed to form the equation whose roots are the squares of the differences of the roots (a, β, γ) of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0, \tag{3}$$

we first remove the second term; the resulting equation is

$$y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0;$$

and the required equation is the same as the equation of squared differences of this latter, since the difference of any two roots is unaltered by removing the second term. We can therefore write down the required equation by putting

$$q = \frac{3H}{a_0^2}, \quad r = \frac{G}{a_0^3}$$

in the above. The result is

$$x^3 + \frac{18H}{a_0^2}x^2 + \frac{81H^2}{a_0^4}x + \frac{27}{a_0^6}(G^2 + 4H^3) = 0, \tag{4}$$

which has for roots

$$(\beta - \gamma)^2, \quad (\gamma - \alpha)^2, \quad (\alpha - \beta)^2.$$

The equation (4) can be written in a form free from fractions by multiplying the roots by a_0^2. It becomes then

$$x^3 + 18Hx^2 + 81H^2x + 27(G^2 + 4H^3) = 0, \tag{5}$$

whose roots are

$$a_0^2(\beta - \gamma)^2, \quad a_0^2(\gamma - \alpha)^2, \quad a_0^2(\alpha - \beta)^2.$$

We can write down from this an important function of the roots of the cubic (3), viz. *the product of the squares of the differences*, in terms of the coefficients :—

$$a_0^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27(G^2 + 4H^3). \tag{6}$$

It is evident from the identity of Art. 37 that $G^2 + 4H^3$ contains a_0^2 as a factor. We have in fact

$$G^2 + 4H^3 \equiv a_0^2\{a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2\}.$$

The expression in brackets is called the *discriminant* of the cubic, and is represented by Δ; giving the identities

$$G^2 + 4H^3 \equiv a_0^2\Delta, \quad HI - a_0J \equiv \Delta.$$

Examples.

1. Form the equation of squared differences of the cubic

$$x^3 - 7x + 6 = 0.$$

Ans. $x^3 - 42x^2 + 441x - 400 = 0$.

2. Form the equation of squared differences of

$$x^3 + 6x^2 + 7x + 2 = 0.$$

First remove the second term.

Ans. $x^3 - 30x^2 + 225x - 68 = 0$.

3. Form the equation of squared differences of

$$x^3 + 6x^2 + 9x + 4 = 0.$$

Ans. $x^3 - 18x^2 + 81x = 0$.

4. What conclusion with respect to the roots of the given cubic can be drawn from the form of the resulting equation in the last example?

43. **Criterion of the Nature of the Roots of a Cubic.**—We can from the form of the equation of differences obtained in Art. 42 derive criteria, in terms of the coefficients, of the nature of the roots of the algebraical cubic. For, when the equation (5) of Art. 42 has a negative root, the cubic (3) must have a pair of imaginary roots, in order that the square of their difference should be negative; and when (5) has no negative root, the cubic (3) has all its roots real, since a pair of imaginary roots of (3) would give rise to a negative root of (5).

In what follows it is assumed that the coefficients of the equation are real quantities. Four cases may be distinguished:—

(1). *When $G^2 + 4H^3$ is negative, the roots of the cubic are all real.*—For, to make this negative H must be negative (and $4H^3 > G^2$); the signs of the equation (5) are then alternately positive and negative, and, therefore (Art. 20), (5) has no negative root; and consequently the given cubic has all its roots real.

(2). *When $G^2 + 4H^3$ is positive, the cubic has two imaginary roots.*—For the equation (5) must then have a negative root.

(3). *When $G^2 + 4H^3 = 0$, the cubic has two equal roots.*—For the equation (5) has then one root equal to zero. In this case $\Delta = 0$, it being assumed that a_0 does not vanish. We may say, therefore, that the *vanishing of the discriminant* (Art. 42) *expresses the condition for equal roots.*

(4). *When $G = 0$, and $H = 0$, the cubic has its three roots equal.*—For the roots of (5) are then all equal to zero. These equations may also be expressed, as can be easily seen, in the form

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3},$$

which relations among the coefficients are therefore the *conditions that the cubic should be a perfect cube.*

44. **Equation of Differences in General.**—The general problem of the formation, by the aid of symmetric functions, of the equation whose roots are the differences, or the squares of the differences, of the roots of a given equation, may be treated as follows:—Let the proposed equation be

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots . (x - \alpha_n) = 0.$$

Substituting $x + a_r$ for x, and giving r the values 1, 2, 3, ... n, in succession, we have the equations

$$\left.\begin{array}{l} f(x+a_1) = x(x+a_1-a_2)(x+a_1-a_3)\ldots\ldots(x+a_1-a_n), \\ f(x+a_2) = x(x+a_2-a_1)(x+a_2-a_3)\ldots\ldots(x+a_2-a_n), \\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f(x+a_n) = x(x+a_n-a_1)(x+a_n-a_2)\ldots\ldots(x+c_n-a_{n-1}). \end{array}\right\} \quad (1)$$

Also, employing the expansion of Art. 6, and observing that $f(a_r) = 0$, we find the equation

$$\frac{1}{x}f(x+a_r) = f'(a_r) + \frac{x}{1\,.\,2}f''(a_r) + \frac{x^2}{1\,.\,2\,.\,3}f'''(a_r) + \ldots\ldots + x^{n-1}.$$

Denoting the second side of this equation by $\phi(x, a_r)$, and multiplying both sides of the identities (1), we obtain

$$\phi(x, a_1)\,\phi(x, a_2)\ldots\ldots\phi(x, a_n) = \{x^2-(a_1-a_2)^2\}\{x^2-(a_1-a_3)^2\}\ldots$$
$$\ldots\{x^2-(a_{n-1}-a_n)^2\}.$$

To form the equation of differences, therefore, we can multiply together the n factors $\phi(x, a_1)$, $\phi(x, a_2)$, &c., and substitute for the symmetric functions of the roots which occur in the product their values in terms of the coefficients. Or we may, as already explained in Art. 42, form directly the product of the $\frac{1}{2}n(n-1)$ factors on the right-hand side of the above identity, and express the symmetric functions involved in terms of the coefficients. The roots of the resulting equation of the $n(n-1)^{th}$ degree in x are equal in pairs with opposite signs. Since in this equation x occurs in even powers only, we may substitute y for x^2, and thus obtain the equation of the $\frac{1}{2}n(n-1)^{th}$ degree whose roots are the squared differences.

For equations beyond the third degree the formation of the equation of differences becomes laborious. We shall give the result in the case of the general algebraic equation of the fourth degree in a subsequent chapter.

EXAMPLES.

1. The roots of the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

are α, β, γ; form the equation whose roots are

$$\beta^2 + \gamma^2, \quad \gamma^2 + \alpha^2, \quad \alpha^2 + \beta^2.$$

Ans. $y^3 - 28y^2 + 245y - 650 = 0.$

2. The roots of the cubic

$$x^3 + 2x^2 + 3x + 1 = 0$$

are α, β, γ; form the equation whose roots are

$$\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}, \quad \frac{1}{\gamma^3} + \frac{1}{\alpha^3} - \frac{1}{\beta^3}, \quad \frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}.$$

Ans. $y^3 + 12y^2 - 172y - 2072 = 0.$

3. The roots of the cubic

$$x^3 + qx + r = 0$$

are α, β, γ; form the equation whose roots are

$$\beta^2 + \beta\gamma + \gamma^2, \quad \gamma^2 + \gamma\alpha + \alpha^2, \quad \alpha^2 + \alpha\beta + \beta^2.$$

Ans. $(y + q)^3 = 0.$

4. The roots of the cubic

$$x^3 + px^2 + qx + r = 0$$

being α, β, γ; form the equation whose roots are

$$\beta^2 + \gamma^2 - \alpha^2, \quad \gamma^2 + \alpha^2 - \beta^2, \quad \alpha^2 + \beta^2 - \gamma^2.$$

Ans. $y^3 - (p^2 - 2q)y^2 - (p^4 - 4p^2q + 8pr)y + p^6 - 6p^4q + 8p^3r$
$+ 8p^2q^2 - 16pqr + 8r^2 = 0.$

5. If α, β, γ be the roots of the cubic

$$x^3 - 3(1 + a + a^2)x + 1 + 3a + 3a^2 + 2a^3 = 0;$$

prove that $(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)$ is a rational function of a.

Ans. $\pm 9(1 + a + a^2).$

6. Find the relation between G and H of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

when its roots are so related that $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$, $(\alpha - \beta)^2$ are in arithmetical progression.

Ans. $G^2 + 2H^3 = 0.$

7. If α, β, γ, δ be the roots of

$$c^2x^4 - 2c^2x^3 + 2x - 1 = 0,$$

find the value of

$$(\beta^2 - \gamma^2)^2(\alpha^2 - \delta^2)^2 + (\gamma^2 - \alpha^2)^2(\beta^3 - \delta^2)^2 + (\alpha^2 - \beta^2)^2(\gamma^2 - \delta^2)^2.$$

Ans. 0.

8. Prove that, if

$$\beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta = 0,$$

$$\{(\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2\}^2$$
$$= 18\,\{(\beta^2-\gamma^2)^2(\alpha^2-\delta^2)^2 + (\gamma^2-\alpha^2)^2(\beta^2-\delta^2)^2 + (\alpha^2-\beta^2)^2(\gamma^2-\delta^2)^2\}.$$

9. Solve the equation

$$x^5 - x^4 + 8x^2 - 9x - 15 = 0,$$

which has one root of the form $1 + \alpha\sqrt{-1}$.

Diminish the roots by 1; substitute $\alpha\sqrt{-1}$ for x: we find that α must satisfy $\alpha^4 - 3\alpha^2 - 4 = 0$, and $\alpha^4 - 6\alpha^2 + 8 = 0$; hence $\alpha = \pm 2$. Hence the factor $x^2 - 2x + 5$. The other factors are $(x+1)$ and (x^2-3), as is evident.

10. The roots of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

are α, β, γ; form the equation whose roots are

$$\beta+\gamma, \quad \gamma+\alpha, \quad \alpha+\beta.$$

This question has been already solved in Art. 41. We give here another solution which, although in this particular instance it is not the simplest, will be found convenient in many examples. Let the roots of the given equation be diminished by h. The transformed equation is (Art. 35)

$$a_0y^3 + 3A_1y^2 + 3A_2y + A_3 = 0,$$

whose roots are $\alpha - h$, $\beta - h$, $\gamma - h$. We express the condition that this equation should have two roots equal with opposite signs. This condition is (see Ex. 17, Art. 24)

$$9A_1A_2 - a_0A_3 = 0.$$

This equation is a cubic in h whose roots are

$$\tfrac{1}{2}(\beta+\gamma), \quad \tfrac{1}{2}(\gamma+\alpha), \quad \tfrac{1}{2}(\alpha+\beta);$$

for the above condition is

$$(\beta - h) + (\gamma - h) = 0,$$

or

$$2h = \beta + \gamma,$$

where β, γ represent indifferently any two of the roots. From the equation in h the required cubic can be formed by multiplying the roots by 2.

11. The roots of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

are $\alpha, \beta, \gamma, \delta$; form the sextic whose roots are

$$\beta+\gamma, \quad \gamma+\alpha, \quad \alpha+\beta, \quad \alpha+\delta, \quad \beta+\delta, \quad \gamma+\delta.$$

Employing the method of Ex. 10, the required equation can be obtained from the condition of Ex. 20, Art. 24.

The condition is in this case

$$6A_1A_2A_3 - A_1^2A_4 - a_0A_3^2 = 0.$$

This is a sextic in h whose roots are $\tfrac{1}{2}(\beta+\gamma)$, &c., from which the required equation can be obtained as in the last example.

12. Form, for the cubic of Ex. 10, the equation whose roots are

$$\frac{\beta\gamma - \alpha^2}{\beta + \gamma - 2\alpha}, \quad \frac{\gamma\alpha - \beta^2}{\gamma + \alpha - 2\beta}, \quad \frac{\alpha\beta - \gamma^2}{\alpha + \beta - 2\gamma}.$$

Diminish the roots by h, and express the condition that the resulting cubic should have its roots in geometric progression (see Ex. 18, Art. 24). The condition is

$$A_1^3 A_3 - a_0 A_2^3 = 0.$$

This will be found to reduce to a cubic in h; whose roots are the values above written, since

$$(\alpha - h)^2 = (\beta - h)(\gamma - h), \quad \text{or} \quad h = \frac{\beta\gamma - \alpha^2}{\beta + \gamma - 2\alpha}.$$

13. Form for the same cubic the equation whose roots are

$$\frac{2\beta\gamma - \alpha\beta - \alpha\gamma}{\beta + \gamma - 2\alpha}, \quad \frac{2\gamma\alpha - \beta\gamma - \beta\alpha}{\gamma + \alpha - 2\beta}, \quad \frac{2\alpha\beta - \gamma\alpha - \gamma\beta}{\alpha + \beta - 2\gamma}.$$

Diminish the roots by h, and express the condition that the transformed cubic should have its roots in harmonic progression (see Ex. 19, Art. 24). We have

$$\frac{2}{\alpha - h} = \frac{1}{\beta - h} + \frac{1}{\gamma - h},$$

or

$$h = \frac{2\beta\gamma - \alpha\beta - \alpha\gamma}{\beta + \gamma - 2\alpha}.$$

The equation in h is

$$a_0 A_3^2 - 3A_1A_2A_3 + 2A_2^3 = 0,$$

which will be found to reduce to a cubic.

14. The roots of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

are α, β, γ, δ; find the cubic whose roots are

$$\frac{\beta\gamma - \alpha\delta}{\beta + \gamma - \alpha - \delta}, \quad \frac{\gamma\alpha - \beta\delta}{\gamma + \alpha - \beta - \delta}, \quad \frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta}.$$

Diminish the roots by h, and employ the condition of Ex. 22, Art. 24. The condition is in this case

$$A_1^2 A_4 - a_0 A_3^2 = 0,$$

which reduces to a cubic in h whose roots are the values above written.

15. Find the equation whose roots are the ratios of the roots of the cubic

$$x^3 + qx + r = 0.$$

The general problem can be solved by elimination. Let $f(x) = 0$ be the given equation, and $\rho = \dfrac{\beta}{\alpha}$ = the ratio of two roots; then since $f(\beta) = 0$, we have $f(\rho\alpha) = 0$, also $f(\alpha) = 0$; and the required equation in ρ is obtained by eliminating

α between these two latter equations. For the cubic in the present example the result is

$$r^2(\rho^2+\rho+1)^3+q^3\rho^2(\rho+1)^2=0.$$

16. If α, β, γ be the roots of

$$x^3+px^2+qx+r=0,$$

form the equation whose roots are

$$\beta^2+\gamma^2, \quad \gamma^2+\alpha^2, \quad \alpha^2+\beta^2.$$

Ans. $x^3-2(p^2-2q)x^2+(p^4-4p^2q+5q^2-2pr)x-(p^2q^2-2p^3r+4pqr-2q^3-r^2)=0.$

17. Form for the same cubic the equation whose roots are

$$\frac{\beta}{\gamma}+\frac{\gamma}{\beta}, \quad \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}, \quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}.$$

Ans. $r^2x^3-(pqr-3r^2)x^2+(p^3r-5pqr+3r^2+q^3)x$
$-(p^2q^2-2p^3r+4pqr-2q^3-r^2)=0.$

18. If α, β, γ be the roots of the cubic

$$x^3+qx+r=0,$$

form the equation whose roots are

$$l\alpha+m\beta\gamma, \quad l\beta+m\gamma\alpha, \quad l\gamma+m\alpha\beta.$$

Ans. $y^3-mqy^2+(l^2q+3lmr)y+l^3r-l^2mq^2-2lm^2qr-m^3r^2=0.$

19. If α, β, γ be the roots of the cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

form the equation whose roots are

$$(\alpha-\beta)(\alpha-\gamma), \quad (\beta-\gamma)(\beta-\alpha), \quad (\gamma-\alpha)(\gamma-\beta).$$

Ans. $y^3+\dfrac{9H}{a_0^2}y^2-\dfrac{27(G^2+4H^3)}{a_0^6}=0.$

20. Form, for the cubic of Ex. 19, the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2, \quad (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2, \quad (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2.$$

The required equation can be obtained by forming the equation of squared differences of the cubic (4) of Art. 42, since

$$(\gamma-\alpha)^2-(\alpha-\beta)^2=(\beta-\gamma)(2\alpha-\beta-\gamma).$$

21. Form, for the cubic of Ex. 16, the equation whose roots are

$$\alpha(\beta-\gamma)^2, \quad \beta(\gamma-\alpha)^2, \quad \gamma(\alpha-\beta)^2.$$

Let the transformed equation be $x^3+Px^2+Qx+R=0$.

Ans. $P=pq-9r, \quad Q=q^3-9pqr+27r^2+p^3r,$
$R=-r(4q^3+27r^2+4p^3r-p^2q^2-18pqr).$

22. Form, for the same cubic, the equation whose roots are

$$\alpha^2+2\beta\gamma, \quad \beta^2+2\gamma\alpha, \quad \gamma^2+2\alpha\beta.$$

Ans. $P=-p^2, \quad Q=q(2p^2-3q), \quad -R=4p^3r-18pqr+2q^3+27r^2.$

CHAPTER V.

SOLUTION OF RECIPROCAL AND BINOMIAL EQUATIONS.

45. **Reciprocal Equations.**—It has been shown in Art. 32 that all reciprocal equations can be reduced to a standard form, in which the degree is even, and the coefficients counting from the beginning and end equal with the same sign. We now proceed to prove that *a reciprocal equation of the standard form can always be depressed to another of half the dimensions.*

Consider the equation

$$a_0 x^{2m} + a_1 x^{2m-1} + \ldots + a_m x^m + \ldots + a_1 x + a_0 = 0.$$

Dividing by x^m, and uniting terms equally distant from the extremes, we have

$$a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots + a_{m-1}\left(x + \frac{1}{x}\right) + a_m = 0.$$

Assume $x + \dfrac{1}{x} = z$, and let $x^p + \dfrac{1}{x^p}$ be denoted for brevity by V_p. We have plainly the relation

$$V_{p+1} = V_p z - V_{p-1}.$$

Giving p in succession the values 1, 2, 3, &c., we have

$$V_2 = V_1 z - V_0 = z^2 - 2,$$

$$V_3 = V_2 z - V_1 = z^3 - 3z,$$

$$V_4 = V_3 z - V_2 = z^4 - 4z^2 + 2,$$

$$V_5 = V_4 z - V_3 = z^5 - 5z^3 + 5z;$$

and so on. Substituting these values in the above equation, we get an equation of the m^{th} degree in z; and from the values of z those of x can be obtained by solving a quadratic.

Examples.

1. Find the roots of the equation

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

Dividing by $x + 1$ (see Art. 32), we have

$$x^4 + x^2 + 1 = 0.$$

This equation may be depressed to the form

$$z^2 - 1 = 0, \quad \text{giving } z = \pm 1;$$

whence $$x + \frac{1}{x} = 1, \quad x + \frac{1}{x} = -1,$$

and the roots of these equations are

$$\frac{1 \pm \sqrt{-3}}{2}, \quad \frac{-1 \pm \sqrt{-3}}{2}.$$

2. Find the roots of the equation

$$x^{10} - 3x^8 + 5x^6 - 5x^4 + 3x^2 - 1 = 0.$$

Dividing by $x^2 - 1$, which may be done briefly as follows (see Art. 8),

$$\begin{array}{cccccc} 1 & -3 & 5 & -5 & 3 & -1 \\ & 1 & -2 & 3 & -2 & 1 \\ \hline & -2 & 3 & -2 & 1 & 0, \end{array}$$

we have the reciprocal equation

$$x^8 - 2x^6 + 3x^4 - 2x^2 + 1 = 0, \tag{1}$$

or $$\left(x^4 + \frac{1}{x^4}\right) - 2\left(x^2 + \frac{1}{x^2}\right) + 3 = 0.$$

Substituting for V_4, $z^4 - 4z^2 + 2$; and for V_2, $z^2 - 2$, we have the equation

$$z^4 - 6z^2 + 9 = 0, \quad \text{or} \quad (z^2 - 3)^2 = 0,$$

whence $$z^2 = 3, \quad \text{and} \quad z = \pm\sqrt{3},$$

giving $$x + \frac{1}{x} = \sqrt{3}, \quad x + \frac{1}{x} = -\sqrt{3};$$

and the roots of these equations are

$$\frac{\sqrt{3} \pm \sqrt{-1}}{2}, \quad \frac{-\sqrt{3} \pm \sqrt{-1}}{2}.$$

These roots are double roots of the equation (1).

3. Solve the equation

$$x^5 - 1 = 0.$$

Dividing by $x - 1$ we have

$$x^4 + x^3 + x^2 + x + 1 = 0;$$

from which we obtain

$$z^2 + z - 1 = 0.$$

Solving this equation, we have the quadratics

$$x^2 + \tfrac{1}{2}(1 + \sqrt{5})x + 1 = 0,$$

$$x^2 + \tfrac{1}{2}(1 - \sqrt{5})x + 1 = 0,$$

from which we obtain

$$x = \tfrac{1}{4}\{-1 + \theta\sqrt{5} \pm (10 + 2\theta\sqrt{5})^{\frac{1}{2}}\sqrt{-1}\},$$

where $\theta^2 = 1$.

This expression gives the four values of x.

4. Find the quadratic factors of

$$x^6 + 1 = 0.$$

Transforming this, we have

$$z^3 - 3z = 0,$$

whence $\qquad z = 0, \quad \text{and} \quad z = \pm\sqrt{3}.$

The quadratic factors of the given equation are, therefore,

$$x^2 + 1 = 0, \quad x^2 \pm \sqrt{3}x + 1 = 0.$$

5. Solve the equations

(1). $(1 + x)^4 = a(1 + x^4)$, $\qquad$ (2). $(1 + x)^5 = a(1 + x^5)$.

6. Reduce to an equation of the fourth degree in z

$$\frac{(1 + x)^5}{1 + x^5} + \frac{(1 - x)^5}{1 - x^5} = 2a.$$

Ans. $(1 - a)z^4 + (7 + 3a)z^2 - (4 + a) = 0$.

46. **Binomial Equations. General Properties.**—In this and the following articles will be proved the leading general properties of binomial equations.

Prop. I.—*If* a *be an imaginary root of* $x^n - 1 = 0$, *then* a^m *also will be a root, m being any integer.*

Since a is a root,

$$a^n = 1, \text{ and therefore } (a^n)^m = 1, \text{ or } (a^m)^n = 1;$$

that is, $\qquad a^m$ is a root of $x^n - 1 = 0$.

The same is true of the equation $x^n + 1 = 0$, except that in this case m must be an *odd* integer.

47. Prop. II.—*If m and n be prime to each other, the equations* $x^m - 1 = 0$, $x^n - 1 = 0$ *have no common root except unity.*

To prove this we make use of the following property of numbers:—*If m and n be integers prime to each other, integers a and b can be found such that $mb - na = \pm 1$.* For, in fact, when $\frac{m}{n}$ is turned into a continued fraction, $\frac{a}{b}$ is the approximation preceding the final restoration of $\frac{m}{n}$.

Now, if possible, let α be any common root of the given equations; then

$$\alpha^m = 1, \text{ and } \alpha^n = 1;$$

therefore $$\alpha^{mb} = 1, \text{ and } \alpha^{na} = 1;$$

whence $$\alpha^{(mb-na)} = 1, \text{ or } \alpha^{\pm 1} = 1, \text{ or } \alpha = 1;$$

that is, 1 is the only root common to the given equations.

48. Prop. III.—*If k be the greatest common measure of two integers m and n, the roots common to the equations $x^m - 1 = 0$, and $x^n - 1 = 0$, are roots of the equation $x^k - 1 = 0$.*

To prove this, let

$$m = km', \quad n = kn'.$$

Now, since m' and n' are prime to each other, integers b and a may be found such that $m'b - n'a = \pm 1$; hence

$$mb - na = \pm k.$$

If, therefore, α be a common root of $x^m - 1 = 0$, and $x^n - 1 = 0$,

$$\alpha^{(mb-na)} = 1, \text{ or } \alpha^k = 1;$$

which proves that α is a root of the equation $x^k - 1 = 0$.

49. Prop. IV.—*When n is a prime number, and α any imaginary root of $x^n - 1 = 0$, all the roots are included in the series*

$$1, \ \alpha, \ \alpha^2, \ldots \alpha^{n-1}.$$

For, by Prop. I., these quantities are all roots of the equation. And they are all different; for, if possible, let any two of them be equal, $\alpha^p = \alpha^q$,

whence $$\alpha^{(p-q)} = 1;$$

but, by Prop. II., this equation is impossible, since n is necessarily prime to $(p - q)$, which is a number less than n.

50. Prop. V.—*When n is a composite number formed of the factors p, q, r, &c., the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$, &c., all satisfy the equation $x^n - 1 = 0$.*

For, consider a root a of the equation $x^p - 1 = 0$; then $a^p = 1$; from which we derive

$$(a^p)^{qr}\ldots = 1; \text{ or } a^n - 1 = 0;$$

which proves the proposition.

51. Prop. VI.—*When n is a composite number formed of the prime factors p, q, r, &c., the roots of the equation $x^n - 1 = 0$ are the n terms of the product*

$$(1 + a + a^2 + \ldots + a^{p-1})(1 + \beta + \ldots + \beta^{q-1})(1 + \gamma + \ldots + \gamma^{r-1})\ldots,$$

where a is a root of $x^p - 1 = 0$, β of $x^q - 1 = 0$, γ of $x^r - 1 = 0$, &c.

We prove this for the case of three factors p, q, r. A similar proof applies in general. Any term, *e.g.* $a^a\beta^b\gamma^c$, of the product is evidently a root of the equation $x^n - 1 = 0$, since $a^{an} = 1$, $\beta^{bn} = 1$, $\gamma^{cn} = 1$, and, therefore, $(a^a\beta^b\gamma^c)^n = 1$. And no two terms of the product can be equal; for, if possible let $a^a\beta^b\gamma^c$ be equal to another term $a^{a'}\beta^{b'}\gamma^{c'}$; then $a^{a'-a} = \beta^{b-b'}\gamma^{c-c'}$. The first member of this equation is a root of $x^p - 1 = 0$, and the second member is a root of $x^{qr} - 1 = 0$. Now these two equations cannot have a common root since p and qr are prime to each other (Prop. II.); hence $a^a\beta^b\gamma^c$ cannot be equal to $a^{a'}\beta^{b'}\gamma^{c'}$.

52. Prop. VII.—*The roots of the equation $x^n - 1 = 0$, where $n = p^aq^br^c$, and p, q, r are the prime factors of n, are the n products of the form $a\beta\gamma$, where a is a root of $x^{p^a} = 1$, β a root of $x^{q^b} = 1$, and γ of $x^{r^c} = 1$.*

This is an extension of Prop. VI. to the case where the prime factors occur more than once in n. The proof is exactly similar. Any such product $a\beta\gamma$ must be a root, since $a^n = 1$, $\beta^n = 1$, $\gamma^n = 1$, n being a multiple of p^a, q^b, r^c; and a proof similar to that of Art. 51 shows that no two such products can be equal, since p^a, q^b, r^c are prime to one another. We have, for convenience, stated this proposition for three factors only of n. A similar proof can be applied to the general case.

From this and the preceding propositions we are now able to derive the following general conclusion:—

The determination of the n^{th} roots of unity is reduced to the case where n is a prime number, or a power of a prime number.

53. **The Special Roots of the Equation** $x^n - 1 = 0$.— Every equation $x^n - 1 = 0$ has certain roots which do not belong to any equation of similar form and lower degree. Such roots we call *special roots** of that equation, or *special n^{th} roots of unity.* If n be a prime number, all the imaginary roots are roots of this kind. If $n = p^a$, where p is a prime number, any n^{th} root of a lower degree than n must belong to the equation $x^{p^{a-1}} - 1 = 0$, since every divisor of p^a is a divisor of p^{a-1} (except n itself); hence there are $p^a\left(1 - \frac{1}{p}\right)$ roots which belong to no lower degree. If, again, $n = p^a q^b$, where p and q are prime to each other, there are $p^a\left(1 - \frac{1}{p}\right)$, and $q^b\left(1 - \frac{1}{q}\right)$ special roots of $x^{p^a} - 1 = 0$, and $x^{q^b} - 1 = 0$, respectively. Now, if a and β be any two special roots of these equations, $a\beta$ is a special root of $x^n - 1 = 0$; for if not, suppose $(a\beta)^m = 1$, where m is less than n; we have then $a^m = \beta^{-m}$; but a^m is a root of $x^{p^a} - 1 = 0$, and β^{-m} is a root of $x^{q^b} - 1 = 0$, and these equations cannot have a common root other than 1, as their degrees are prime to each other; consequently m cannot be less than n, and $a\beta$ is a special root of $x^n - 1 = 0$. Also, as there are

$$p^a\left(1 - \frac{1}{p}\right)q^b\left(1 - \frac{1}{q}\right), \text{ or } n\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right),$$

such products, there are the same number of special n^{th} roots. This proof may be extended without difficulty to any form of n.

All the roots of $x^n - 1 = 0$ *are given by the series* $1, a, a^2, \ldots a^{n-1}$; *where* a *is any special n^{th} root.* For it is plain that a, a^2, &c., are all roots. And no two are equal; for, if $a^p = a^q$, $a^{(p-q)} = 1$; and therefore a is not a special n^{th} root, since $p - q$ is less than n.

When one special n^{th} root a is given, we may obtain all the other special n^{th} roots of unity.

* The term "special root" is here used in preference to the usual term "primitive root," since the latter has a different signification in the theory of numbers.

Since a is a special root, all the roots $1, a, a^2, \ldots a^{n-1}$ are different n^{th} roots, as we have just proved; and if we select a root a^p of this series, where p is prime to n, the roots

$$a^p, a^{2p}, \ldots a^{(n-1)p}, a^{np} (=1)$$

are all different, since the exponents of a when divided by n give different remainders in every case; that is, the series of numbers $0, 1, 2, 3, \ldots n-1$ in some order; whence this series of roots is the same as the former, except that the terms occur in a different order. To each number p, prime to n and less than it (1 included), corresponds a special n^{th} root of unity; for a^{mp} cannot be equal to 1 when m is less than n, for if it were we should have two roots in the series equal to 1, and the series could not give all the roots in that case; therefore a^p is not a root of any binomial equation of a degree inferior to n; that is, a^p is a special n^{th} root of unity. What is here proved agrees with the result above established, since the number of integers less than n and prime to it is, by a known property of numbers, $n\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)$ when $n = p^a q^b$, which is also, as above proved, the number of special roots of $x^n - 1 = 0$.

Examples.

1. To determine the special roots of $x^6 - 1 = 0$.

Here, $6 = 2 \times 3$. Consequently the roots of the equations $x^2 - 1 = 0$, and $x^3 - 1 = 0$ are roots of $x^6 - 1 = 0$. Now, dividing $x^6 - 1$ by $x^3 - 1$ we have $x^3 + 1$; and dividing $x^3 + 1$ by $\dfrac{x^2-1}{x-1}$, or $x + 1$, we have $x^2 - x + 1 = 0$, which determines the special roots of $x^6 - 1 = 0$.

Solving this quadratic, the roots are

$$a = \frac{1+\sqrt{-3}}{2}, \quad a_1 = \frac{1-\sqrt{-3}}{2};$$

also since

$$a a_1 = 1 = a^6,$$

$$a_1 = a^5,$$

which may be easily verified.

The special roots are, therefore,

$$a, a^5; \text{ or } a_1^5, a_1; \text{ or } a, \frac{1}{a}.$$

2. To discuss the special roots of $x^{12} - 1 = 0$.

Since 2 and 3 are the prime factors of 12, and $\frac{12}{2} = 6$, $\frac{12}{3} = 4$, the roots of $x^6 - 1 = 0$, and $x^4 - 1 = 0$, are roots of $x^{12} - 1 = 0$; now, dividing $x^{12} - 1$ by $x^4 - 1$, and $x^6 - 1$, and equating the quotients to zero, we have the two equations $x^8 + x^4 + 1 = 0$, and $x^6 + 1 = 0$, both of which must be satisfied by the special roots of $x^{12} - 1 = 0$; therefore, taking the greatest common measure of $x^8 + x^4 + 1$, and $x^6 + 1$, and equating it to zero, the special roots are the roots of the equation $x^4 - x^2 + 1 = 0$.

The same result would plainly have been arrived at by dividing $x^{12} - 1$ by the least common multiple of $x^4 - 1$ and $x^6 - 1$. Now, solving the reciprocal equation $x^4 - x^2 + 1 = 0$, we have $x + \frac{1}{x} = \pm \sqrt{3}$; whence, if α and α_1 be two special roots

$$\left(\alpha, \frac{1}{\alpha}\right) = \frac{\sqrt{3} \pm \sqrt{-1}}{2}, \quad \left(\alpha_1, \frac{1}{\alpha_1}\right) = \frac{-\sqrt{3} \pm \sqrt{-1}}{2}$$

are the four special roots of $x^{12} - 1 = 0$.

We proceed now to express the four special roots in terms of any one of them α.

Since $$\alpha + \frac{1}{\alpha} + \alpha_1 + \frac{1}{\alpha_1} = 0, \text{ or } (\alpha + \alpha_1)\left(1 + \frac{1}{\alpha\alpha_1}\right) = 0,$$

we take $\alpha\alpha_1 = -1$ (as consistent with the values we have assigned to α and α_1); and since α and α_1 are roots of $x^6 + 1 = 0$, $\alpha^6 = -1$, and $\alpha^5 = -\frac{1}{\alpha} = \alpha_1$. The roots $\alpha, \alpha_1, \frac{1}{\alpha_1}, \frac{1}{\alpha}$ may therefore be expressed by the series $\alpha, \alpha^5, \alpha^7, \alpha^{11}$, since $\alpha^{12} = 1$.

Further, replacing α by $\alpha^5, \alpha^7, \alpha^{11}$, we have, including the series just determined, the four following series, by omitting multiples of 12 in the exponents of α:—

$$\begin{array}{cccc}
\alpha, & \alpha^5, & \alpha^7, & \alpha^{11}, \\
\alpha^5, & \alpha, & \alpha^{11}, & \alpha^7, \\
\alpha^7, & \alpha^{11}, & \alpha, & \alpha^5, \\
\alpha^{11}, & \alpha^7, & \alpha^5, & \alpha,
\end{array}$$

where the same roots are reproduced in every row and column, their order only being changed. We have therefore proved that this property is not peculiar to any one root of the four special roots; and it will be noticed, in accordance with what is above proved in general, that 1, 5, 7, and 11 are all the numbers prime to 12, and less than it. We may obtain all the roots of $x^{12} - 1 = 0$ by the powers of any one of the four special roots $\alpha, \alpha^5, \alpha^7, \alpha^{11}$, as follows:—

$$\begin{array}{cccccccccccc}
\alpha, & \alpha^2, & \alpha^3, & \alpha^4, & \alpha^5, & \alpha^6, & \alpha^7, & \alpha^8, & \alpha^9, & \alpha^{10}, & \alpha^{11}, & 1, \\
\alpha^5, & \alpha^{10}, & \alpha^3, & \alpha^8, & \alpha, & \alpha^6, & \alpha^{11}, & \alpha^4, & \alpha^9, & \alpha^2, & \alpha^7, & 1, \\
\alpha^7, & \alpha^2, & \alpha^9, & \alpha^4, & \alpha^{11}, & \alpha^6, & \alpha, & \alpha^8, & \alpha^3, & \alpha^{10}, & \alpha^5, & 1, \\
\alpha^{11}, & \alpha^{10}, & \alpha^9, & \alpha^8, & \alpha^7, & \alpha^6, & \alpha^5, & \alpha^4, & \alpha^3, & \alpha^2, & \alpha, & 1.
\end{array}$$

3. Prove that the special roots of $x^{15} - 1 = 0$ are roots of the equation

$$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = 0.$$

4. Show that the eight roots of the equation in the preceding example may be obtained by multiplying the two roots of $x^2 + x + 1 = 0$ by the four roots of

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

5. Form the equation of the 12th degree whose roots are the special roots of $x^{21} - 1 = 0$, and reduce it to one of half the dimensions.

Ans. $x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 = 0.$

54. **Solution of Binomial Equations by Circular Functions.**—We take the most general binomial equation

$$x^n = a + b\sqrt{-1},$$

where a and b are real quantities.

Let $$a = R\cos\alpha, \quad b = R\sin\alpha;$$

then $$x^n = R(\cos\alpha + \sqrt{-1}\sin\alpha);$$

now, if $$r(\cos\theta + \sqrt{-1}\sin\theta)$$

be a root of this equation, we have, by De Moivre's Theorem,

$$r^n(\cos n\theta + \sqrt{-1}\sin n\theta) = R(\cos\alpha - \sqrt{-1}\sin\alpha);$$

and, therefore,

$$r^n \cos n\theta = R\cos\alpha,$$

$$r^n \sin n\theta = R\sin\alpha.$$

Squaring these two equalities, and adding,

$$r^{2n} = R^2, \text{ giving } r^n = R;$$

where we take R and r both positive, since in expressions of the kind here considered the factor containing the angle may always be taken to involve the sign.

We have then

$$\cos n\theta = \cos\alpha, \quad \sin n\theta = \sin\alpha;$$

and, consequently,

$$n\theta = \alpha + 2k\pi,$$

k being any integer; whence the assumed n^{th} root is of the

general type

$$\sqrt[n]{R}\left(\cos\frac{a+2k\pi}{n}+\sqrt{-1}\sin\frac{a+2k\pi}{n}\right).$$

Giving to k in this expression any n consecutive values in the series of numbers between $-\infty$ and $+\infty$, we get all the n^{th} roots; and no more than n, since the n values recur in periods.

We may write the expression for the n^{th} root under the form

$$\left\{\sqrt[n]{R}\left(\cos\frac{a}{n}+\sqrt{-1}\sin\frac{a}{n}\right)\right\}\left(\cos\frac{2k\pi}{n}+\sqrt{-1}\sin\frac{2k\pi}{n}\right).$$

If we now suppose $R = 1$, and $a = 0$, the equation $x^n = a + b\sqrt{-1}$ becomes $x^n = 1 + 0\sqrt{-1}$; the general type, therefore, of an n^{th} root of $1 + 0\sqrt{-1}$, or unity, is

$$\cos\frac{2k\pi}{n}+\sqrt{-1}\sin\frac{2k\pi}{n}.$$

If we give k any definite value, for instance zero,

$$\sqrt[n]{R}\left(\cos\frac{a}{n}+\sqrt{-1}\sin\frac{a}{n}\right)$$

is one n^{th} root of $a + b\sqrt{-1}$.

The preceding formula shows, therefore, that *all the n^{th} roots of any imaginary quantity may be obtained by multiplying any one of them by the n^{th} roots of unity.*

Taking in conjunction the binomial equations

$$x^n = a + b\sqrt{-1}, \text{ and } x^n = a - b\sqrt{-1},$$

we see that the factors of the trinomial

$$x^{2n} - 2R\cos a \,.\, x^n + R^2$$

are

$$\sqrt[n]{R}\left\{\cos\frac{a+2k\pi}{n}\pm\sqrt{-1}\sin\frac{a+2k\pi}{n}\right\},$$

where k has the values $0, 1, 2, 3 \ldots n-1$.

Examples.

1. Solve the equation $x^7 - 1 = 0$.

Dividing by $x - 1$, this is reduced to the standard form of reciprocal equation. Assuming $z = x + \frac{1}{x}$, we obtain the cubic

$$z^3 + z^2 - 2z - 1 = 0,$$

from whose solution that of the required equation is obtained.

2. Resolve $(x + 1)^7 - x^7 - 1$ into factors.

Ans. $7x(x + 1)(x^2 + x + 1)^2$.

3. Find the quintic on whose solution that of the binomial equation $x^{11} - 1 = 0$ depends.

Ans. $z^5 + z^4 - 4z^3 - 3z^2 + 3z + 1 = 0$.

4. When a binomial equation is reduced to the standard form of reciprocal equation (by division by $x - 1$, $x + 1$, or $x^2 - 1$), show that the reduced equation has all its roots imaginary. (Cf. Examples 15, 16, p. 33.)

5. When this reduced reciprocal equation is transformed by the substitution $z = x + \frac{1}{x}$; show that the equation in z has all its roots real, and situated between -2 and 2.

For the roots of the equation in x are of the form $\cos\alpha + \sqrt{-1}\sin\alpha$ (see Art. 54); hence $x + \frac{1}{x}$ is of the form $2\cos\alpha$, and the value of this is real and between -2 and 2.

6. Show that the following equation is reciprocal, and solve it:—

$$4(x^2 - x + 1)^3 - 27x^2(x - 1)^2 = 0.$$

Ans. Roots: $2, 2, \frac{1}{2}, \frac{1}{2}, -1, -1$.

7. Exhibit all the roots of the equation $x^9 - 1 = 0$.

The solution of this is reduced to the solution of the three cubics

$$x^3 - 1 = 0, \quad x^3 - \omega = 0, \quad x^3 - \omega^2 = 0;$$

where ω, ω^2 are the imaginary cube roots of unity. The nine roots may be represented as follows:—

$$1,\ \omega^{\frac{1}{3}},\ \omega^{\frac{2}{3}},\ \omega,\ \omega^{\frac{4}{3}},\ \omega^{\frac{5}{3}},\ \omega^2,\ \omega^{\frac{7}{3}},\ \omega^{\frac{8}{3}}.$$

Excluding $1, \omega, \omega^2$; the other six roots are special roots of the given equation; and are the roots of the sextic

$$x^6 + x^3 + 1 = 0.$$

8. Reducing the equation of the 8^{th} degree in Ex. 3, Art. 53, by the substitution $z = x + \frac{1}{x}$, we obtain

$$z^4 - z^3 - 4z^2 + 4z + 1 = 0;$$

prove that the roots of this equation are

$$2\cos\frac{2\pi}{15},\quad 2\cos\frac{4\pi}{15},\quad 2\cos\frac{8\pi}{15},\quad 2\cos\frac{14\pi}{15}.$$

9. Reduce the equation

$$4x^4 - 85x^3 + 357x^2 - 340x + 64 = 0$$

to a reciprocal equation, and solve it.

Assume $$z = \frac{x}{2} + \frac{2}{x}.$$ *Ans.* Roots: $\frac{1}{4}$, 1, 4, 16.

10. Solve the equation

$$x^4 + mpx^3 + m^2qx^2 + m^3px + m^4 = 0.$$

Dividing the roots by m, this reduces to a reciprocal equation.

11. If a be an imaginary root of the equation $x^n - 1 = 0$, where n is a prime number; prove the relation

$$(1-a)(1-a^2)(1-a^3) \ldots (1-a^{n-1}) = n.$$

12. Show that a cubic equation can be reduced immediately to the reciprocal form when the relation of Ex. 18, Art. 24, exists amongst its coefficients.

13. Show that a biquadratic can be reduced immediately to the reciprocal form when the relation of Ex. 22, Art. 24, exists amongst its coefficients.

14. Form the cubic whose roots are

$$a + a^6,\quad a^3 + a^4,\quad a^2 + a^5,$$

where a is an imaginary root of $x^7 - 1 = 0$. *Ans.* $x^3 + x^2 - 2x - 1 = 0$.

When the roots of this cubic are known, the solution of the equation $x^7 - 1 = 0$ may be completed by means of quadratics. For, suppose the three roots to be x_1, x_2, x_3; then a and a^6 are the roots of $x^2 - x_1x + 1 = 0$; a^3 and a^4 of $x^2 - x_2x + 1 = 0$, and a^2 and a^5 of $x^2 - x_3x + 1 = 0$. It is easy to see that the roots of the cubic are all real, and they may be readily found approximately by the methods of Chap. X.

15. Form the cubic whose roots are

$$a + a^8 + a^{12} + a^5,\quad a^2 + a^3 + a^{11} + a^{10},\quad a^4 + a^6 + a^9 + a^7,$$

where a is an imaginary root of $x^{13} - 1 = 0$. *Ans.* $x^3 + x^2 - 4x + 1 = 0$.

As in the preceding example, when the roots of the cubic (which are all real) are known, the solution of the binomial equation $x^{13} - 1 = 0$ may be completed by solving quadratics. Let x_1, x_2, x_3 be the roots of the cubic. It is easily seen that $a + a^{12}$ and $a^8 + a^5$ are the roots of $x^2 - x_1x + x_3 = 0$; $a^2 + a^{11}$ and $a^3 + a^{10}$ of $x^2 - x_2x + x_1 = 0$, and $a^4 + a^9$ and $a^6 + a^7$ of $x^2 - x_3x + x_2 = 0$. When these quadratics are solved, each pair of roots a, a^{12}; a^8, a^5, &c., may be found by the solution of another quadratic, as in the preceding example.

16. Reduce to quadratics the solution of $x^{17} - 1 = 0$.

Calling α one of the imaginary roots, we form the quadratic whose roots are

$$\alpha_1 \equiv \alpha + \alpha^9 + \alpha^{13} + \alpha^{15} + \alpha^{16} + \alpha^8 + \alpha^4 + \alpha^2,$$
$$\alpha_2 \equiv \alpha^3 + \alpha^{10} + \alpha^5 + \alpha^{11} + \alpha^{14} + \alpha^7 + \alpha^{12} + \alpha^6.$$

We easily find $\alpha_1\alpha_2 = 4(\alpha_1 + \alpha_2) = -4$; hence α_1 and α_2 are the roots of $x^2 + x - 4 = 0$, and may be found by solving this quadratic. Assuming, again,

$$\left.\begin{array}{l}\beta_1 \equiv \alpha + \alpha^{13} + \alpha^{16} + \alpha^4, \\ \beta_2 \equiv \alpha^9 + \alpha^{15} + \alpha^8 + \alpha^2,\end{array}\right\} \qquad \left.\begin{array}{l}\gamma_1 \equiv \alpha^3 + \alpha^5 + \alpha^{14} + \alpha^{12}, \\ \gamma_2 \equiv \alpha^{10} + \alpha^{11} + \alpha^7 + \alpha^6,\end{array}\right\}$$

it is seen that β_1, β_2 are the roots of $x^2 - \alpha_1 x - 1 = 0$, and γ_1, γ_2 of $x^2 - \alpha_2 x - 1 = 0$. Separating again each of these into two parts, and forming the quadratic whose roots are, for example, $\alpha + \alpha^{16}$ and $\alpha^{13} + \alpha^4$, the sums of the roots in pairs are obtained; and finally the roots themselves, by the solution of quadratics, as in the preceding examples.

This and the preceding two are examples of Gauss's method of solving algebraically the binomial equation $x^n - 1 = 0$ when n is a prime number. The solution of such an equation can be made to depend on the solution of equations of degree not higher than the greatest prime number which is a factor in $n - 1$. When $n = 13$, *e.g.* the solution depends on that of a cubic, $n - 1$ being $= 3 \cdot 2^2$ in that case; and when $n = 17$, the solution is reducible to quadratics, $n - 1$ being then $= 2^4$. For the application of Gauss's method it is necessary to arrange the $n - 1$ imaginary roots in a suitable order in each case according to the powers of any one of them. A "primitive root" of a prime number n possesses the property that when raised to successive powers from 0 to $n - 2$ inclusive, and divided in each case by n, the $n - 1$ remainders are all different. (See Serret's *Cours d'Algèbre Supérieure*, vol. II. sect. 3.) There are several such primitive roots of any prime number: *e.g.* 2, 6, 7, and 11 of 13, and 3, 5, 6, 7, 10, 11, 12, 14 of 17. Gauss arranges the imaginary roots so that the successive indices of any one of them, α, are the successive powers from 0 to $n - 2$ of any primitive root of n. Taking, for example, the lowest primitive root of 13, and dividing the successive powers of 2 by 13, we get the following series of remainders—

$$1 \quad 2 \quad 4 \quad 8 \quad 3 \quad 6 \quad 12 \quad 11 \quad 9 \quad 5 \quad 10 \quad 7;$$

and these, therefore, are the successive powers of α in order when the indices which exceed 13 are reduced by the equation $\alpha^{13} = 1$. If the lowest primitive root of 17 be treated in the same way, we get the following series of remainders:—

$$1 \quad 3 \quad 9 \quad 10 \quad 13 \quad 5 \quad 15 \quad 11 \quad 16 \quad 14 \quad 8 \quad 7 \quad 4 \quad 12 \quad 2 \quad 6.$$

On comparing these series with the assumptions above made, it will be observed that in the former case, viz. $n = 13$, the twelve roots were divided into three sums of four each, and in the latter case into two sums of eight each. The method of partition in any case depends on the nature of the factors of $n - 1$; and it is not difficult to show in general that the product of any two such groups is equal to the sum of two or more, as the student will have observed in the particular applications given above.

The lowest primitive root in any particular case is the only one necessary to be known for the application of Gauss's method; and this can usually be found without difficulty by trial. It may be observed that one or other of the three simplest prime numbers 2, 3, 5, is a primitive root in the case of every prime number less than 100, with the exception of 41 and 71, whose lowest primitive roots are 6 and 7 respectively. Methods of finding all the primitive roots are given in the section of Serret's work above referred to.

17. Find by trial the lowest primitive root of 19, and hence show how to solve the equation $x^{19} - 1 = 0$.

It is readily found that 2 is a primitive root, and the remainders after division by 19 are given in the process of trial. Since $18 = 3^2.2$, the solution will be effected by cubics and quadratics. The first cubic is found by forming the equation whose roots are

$$\alpha + \alpha^8 + \alpha^7 + \alpha^{18} + \alpha^{11} + \alpha^{12},$$
$$\alpha^2 + \alpha^{16} + \alpha^{14} + \alpha^{17} + \alpha^3 + \alpha^5,$$
$$\alpha^4 + \alpha^{13} + \alpha^9 + \alpha^{15} + \alpha^6 + \alpha^{10}.$$

18. Show that of binomial equations whose degree is a prime number the lowest after $x^{17} - 1 = 0$ whose solution depends on quadratics is $x^{257} - 1 = 0$.

The next prime number after 257 which satisfies the condition that $n - 1$ is a power of 2 is 65537. We have therefore the series 3, 5, 17, 257, 65537, &c.; and Gauss remarks (*Disquisitiones Arithmeticæ*, Art. 365) that the division of a circle into n equal parts, or the description of a regular polygon of n sides, can be effected by geometrical constructions when n has any of these values.

19. If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0,$$

form the equation whose roots are

$$\alpha_1 + \frac{1}{\alpha_1}, \quad \alpha_2 + \frac{1}{\alpha_2}, \ldots \alpha_n + \frac{1}{\alpha_n}.$$

We have here the identity

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n \equiv (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n);$$

and changing x into $\frac{1}{x}$ (see Art. 32),

$$p_n x^n + p_{n-1} x^{n-1} + \ldots + p_2 x^2 + p_1 x + 1 \equiv p_n \left(x - \frac{1}{\alpha_1}\right)\left(x - \frac{1}{\alpha_2}\right) \ldots \left(x - \frac{1}{\alpha_n}\right).$$

Multiplying together these identities, and dividing by x^n, the factors on the right-hand side take the form $x + \frac{1}{x} - \left(\alpha + \frac{1}{\alpha}\right)$; and assuming $x + \frac{1}{x} = z$, the left-hand side can be expressed as a polynomial of the n^{th} degree in z by means of the relations of Art. 45.

20. Find the value of the symmetric function $\Sigma\alpha^2\beta^2(\gamma - \delta)^2$ of the roots of the equation

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0.$$

This can be derived from the result of Ex. 19, p. 52, by changing the roots into their reciprocals, forming $\Sigma\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$ of the transformed equation, and multiplying by $\alpha^2\beta^2\gamma^2\delta^2$, which is equal to $\frac{a_4^2}{a_0^2}$.

Ans $a_0^2\Sigma\alpha^2\beta^2(\gamma-\delta)^2 = 48(a_3^2-a_2a_4)$.

From the values of the symmetric functions given in Chapter III. several others can be obtained by the process here indicated.

21. Find the value of the symmetric function $\Sigma(\alpha_1-\alpha_2)^2\alpha_3^2\alpha_4^2 \ldots \alpha_n^2$ of the roots of the equation

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1 \cdot 2}a_2x^{n-2} + \ldots + na_{n-1}x + a_n = 0.$$

We easily obtain $a_0^2\Sigma(\alpha_1-\alpha_2)^2 = n^2(n-1)(a_1^2-a_0a_2)$; and changing the roots into their reciprocals we have

$$a_0^2\Sigma(\alpha_1-\alpha_2)^2\alpha_3^2\alpha_4^2 \ldots \alpha_n^2 = n^2(n-1)(a_{n-1}^2-a_{n-2}a_n).$$

22. Show that the five roots of the equation

$$x^5 - 5px^3 + 5p^2x + 2q = 0$$

are

$$\sqrt[5]{a} + \sqrt[5]{b},\quad \theta\sqrt[5]{a} + \theta^4\sqrt[5]{b},\quad \theta^2\sqrt[5]{a} + \theta^3\sqrt[5]{b},$$

$$\theta^4\sqrt[5]{a} + \theta\sqrt[5]{b},\quad \theta^3\sqrt[5]{a} + \theta^2\sqrt[5]{b},$$

where $\sqrt[5]{ab} = p$, $a + b = -2q$, and θ is an imaginary fifth root of unity.

N.B.—A quintic reducible to this form can consequently be immediately solved.

23. Write down trigonometrical expressions for the roots in the preceding example; and, p being supposed essentially positive, prove—

(1) when $p^5 < q^2$, the roots are one real and four imaginary;

(2) when $p^5 > q^2$, the roots are all real;

(3) when $p^5 = q^2$, there is a square quadratic factor.

24. Find the following product, where θ is an imaginary fifth root of unity :—

$$(\alpha+\beta+\gamma)(\alpha+\theta\beta+\theta^4\gamma)(\alpha+\theta^2\beta+\theta^3\gamma)(\alpha+\theta^3\beta+\theta^2\gamma)(\alpha+\theta^4\beta+\theta\gamma).$$

Ans. $\alpha^5+\beta^5+\gamma^5-5\alpha\beta\gamma(\alpha^2-\beta\gamma)$.

25. Form the biquadratic equation whose roots are

$$\alpha+2\alpha^4,\quad \alpha^2+2\alpha^3,\quad \alpha^3+2\alpha^2,\quad \alpha^4+2\alpha,$$

where α is an imaginary root of $x^5-1=0$.

Ans. $x^4+3x^3-x^2-3x+11=0$.

CHAPTER VI.

ALGEBRAIC SOLUTION OF THE CUBIC AND BIQUADRATIC.

55. **On the Algebraic Solution of Equations.**—Before proceeding to the solution of cubic and biquadratic equations we make some introductory remarks, with a view of putting clearly before the student the general principles on which the algebraic solution of these equations depends. With this object we give in the present Article three methods of solution of the quadratic, and state as we proceed how these methods may be extended to cubic and biquadratic equations, leaving to subsequent Articles the complete development of the principles involved.

(1). *First method of solution—by assuming for a root a general form involving radicals.*

Since the expression $p + \sqrt{q}$ has two, and only two, values when the square root involved is taken with the double sign, this is a natural form to take for the root of a quadratic. Assuming, therefore, $x = p + \sqrt{q}$, and rationalizing, we have

$$x^2 - 2px + p^2 - q = 0.$$

Now, if this be identical with a given quadratic equation

$$x^2 + Px + Q = 0,$$

we have $$2p = -P, \quad p^2 - q = Q,$$

giving $$x = p + \sqrt{q} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2},$$

which is the solution of the quadratic.

In the case of the cubic equation we shall find that

$$\sqrt[3]{p} + \frac{A}{\sqrt[3]{p}}, \text{ and } \sqrt[3]{p}\sqrt[3]{q}\,(\sqrt[3]{p} + \sqrt[3]{q})$$

are both proper forms to represent a root, these expressions having each three, and only three, values when the cube roots involved are taken in all generality.

In the case of the biquadratic equation we shall find that

$$\sqrt{p}+\sqrt{q}+\frac{A}{\sqrt{p}\sqrt{q}}, \qquad \sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q}$$

are forms which may represent a root, these expressions each giving four, and only four, values of x when the square roots receive their double signs.

(2). *Second method of solution—by resolving into factors.*

Let it be required to resolve the quadratic x^2+Px+Q into its simple factors. For this purpose we put it under the form

$$x^2+Px+Q+\theta-\theta,$$

and determine θ so that

$$x^2+Px+Q+\theta$$

may be a perfect square, *i.e.* we make

$$\theta+Q=\frac{P^2}{4}, \text{ or } \theta=\frac{P^2-4Q}{4};$$

whence, putting for θ its value, we have

$$x^2+Px+Q\equiv\left(x+\frac{P}{2}\right)^2-\left(0x+\frac{\sqrt{P^2-4Q}}{2}\right)^2.$$

Thus we have reduced the quadratic to the form u^2-v^2; and its simple factors are $u+v$, and $u-v$.

Subsequently we shall reduce the cubic to the form

$$(lx+m)^3-(l'x+m')^3, \quad \text{or} \quad u^3-v^3,$$

and obtain its solution from the simple equations

$$u-v=0, \quad u-\omega v=0, \quad u-\omega^2 v=0.$$

It will be shown also that the biquadratic may be reduced to either of the forms

$$(lx^2+mx+n)^2-(l'x^2+m'x+n')^2,$$

$$(x^2+px+q)(x^2+p'x+q'),$$

by solving a cubic equation; and, consequently, the solution of the biquadratic completed by solving two quadratics, viz. in the first case, $lx^2 + mx + n = \pm (l'x^2 + m'x + n')$; and in the second case, $x^2 + px + q = 0$, and $x^2 + p'x + q' = 0$.

(3). *Third method of solution—by symmetric functions of the roots.*

Consider the quadratic equation $x^2 + Px + Q = 0$, of which the roots are α, β. We have the relations

$$\alpha + \beta = - P,$$
$$\alpha\beta = \quad Q.$$

If we attempt to determine α and β by these equations, we fall back on the original equation (see Art. 24); but if we could obtain a second equation between the roots and coefficients, of the form $l\alpha + m\beta = f(P, Q)$, we could easily find α and β by means of this equation and the equation $\alpha + \beta = - P$.

Now in the case of the quadratic there is no difficulty in finding the required equation; for, obviously,

$$(\alpha - \beta)^2 = P^2 - 4Q; \text{ and, therefore, } \alpha - \beta = \sqrt{P^2 - 4Q}.$$

In the case of the cubic equation $x^3 + Px^2 + Qx + R = 0$, we require *two* simple equations of the form

$$l\alpha + m\beta + n\gamma = f(P, Q, R),$$

in addition to the equation $\alpha + \beta + \gamma = - P$, to determine the roots α, β, γ. It will subsequently be proved that the functions

$$(\alpha + \omega\beta + \omega^2\gamma)^3, \quad (\alpha + \omega^2\beta + \omega\gamma)^3$$

may be expressed in terms of the coefficients by solving a *quadratic* equation; and when their values are known the roots of the cubic may be easily found.

In the case of the biquadratic equation

$$x^4 + Px^3 + Qx^2 + Rx + S = 0$$

we require *three* simple equations of the form

$$l\alpha + m\beta + n\gamma + r\delta = f(P, Q, R, S),$$

in addition to the equation

$$\alpha + \beta + \gamma + \delta = -P,$$

to determine the roots $\alpha, \beta, \gamma, \delta$. It will be proved in Art. 66, that the three functions

$$(\beta + \gamma - \alpha - \delta)^2, \quad (\gamma + \alpha - \beta - \delta)^2, \quad (\alpha + \beta - \gamma - \delta)^2$$

may be expressed in terms of the coefficients by solving a *cubic* equation; and when their values are known the roots of the biquadratic equation may be immediately obtained.

56. **The Algebraic Solution of the Cubic Equation.**—Let the general cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

be put under the form

$$z^3 + 3Hz + G = 0,$$

where

$$z \equiv ax + b, \quad H \equiv ac - b^2, \quad G \equiv a^2d - 3abc + 2b^3 \text{ (Art. 36).}$$

To solve this equation, assume*

$$z = \sqrt[3]{p} + \sqrt[3]{q};$$

hence, cubing,

$$z^3 = p + q + 3\sqrt[3]{p}\sqrt[3]{q}\left(\sqrt[3]{p} + \sqrt[3]{q}\right);$$

therefore

$$z^3 - 3\sqrt[3]{p}\sqrt[3]{q}\,.\,z - (p + q) = 0.$$

Now, comparing coefficients, we have

$$\sqrt[3]{p}\sqrt[3]{q} = -H, \quad p + q = -G;$$

from which equations we obtain

$$p = \tfrac{1}{2}\left(-G + \sqrt{G^2 + 4H^3}\right), \quad q = \tfrac{1}{2}\left(-G - \sqrt{G^2 + 4H^3}\right);$$

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* This solution is usually called *Cardan's solution of the cubic.* See Note A at the end of the volume.

and, substituting for $\sqrt[3]{q}$ its value $\dfrac{-H}{\sqrt[3]{p}}$, we have

$$z = \sqrt[3]{p} + \frac{-H}{\sqrt[3]{p}}$$

as the algebraic solution of the equation

$$z^3 + 3Hz + G = 0.$$

It should be noted that if p be replaced by q this value of z is unchanged, as the terms are then simply interchanged; also, since $\sqrt[3]{p}$ has the three values $\sqrt[3]{p}$, $\omega\sqrt[3]{p}$, $\omega^2\sqrt[3]{p}$, obtained by multiplying any one of its values by the three cube roots of unity, we obtain three, and only three, values for z, namely,

$$\sqrt[3]{p} + \frac{-H}{\sqrt[3]{p}}, \quad \omega\sqrt[3]{p} + \omega^2\frac{-H}{\sqrt[3]{p}}, \quad \omega^2\sqrt[3]{p} + \omega\frac{-H}{\sqrt[3]{p}};$$

the order of these values only changing according to the cube root of p selected.

Now, if z be replaced by its value $ax + b$, we have, finally,

$$ax + b = \sqrt[3]{p} + \frac{-H}{\sqrt[3]{p}}$$

(where p has the value previously determined in terms of the coefficients) as the *complete algebraic solution of the cubic equation*

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

the square root and cube root involved being taken in their entire generality.

57. **Application to Numerical Equations.**—The solution of the cubic which has been obtained, unlike the solution of the quadratic, is of little practical value when the coefficients of the equation are given numbers; although as an algebraic solution it is complete.

For, when the roots of the cubic are all real, $G^2 + 4H^3 = -K^2$, an essentially negative number (see Art. 43); and, substituting for p and q their values

$$\tfrac{1}{2}(-G \pm K\sqrt{-1})$$

in the formula $\sqrt[3]{p} + \sqrt[3]{q}$, we have the following expression for a root of the cubic:—

$$\left(\frac{-G + K\sqrt{-1}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - K\sqrt{-1}}{2}\right)^{\frac{1}{3}}.$$

Now there is no general arithmetical process for extracting the cube root of such complex numbers, and consequently this formula is useless for purposes of arithmetical calculation.

But when the cubic has a pair of imaginary roots, a numerical value may be obtained from the formula

$$\left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}},$$

since $G^2 + 4H^3$ is positive in this case. As a practical method, however, of obtaining the real root of a numerical cubic, this process is of little value.

In the first case, namely, where the roots are all real, we can make use of Trigonometry to obtain the numerical values of the roots in the following manner:—

Assuming $2R \cos \phi = -G$, and $2R \sin \phi = K$,

we have $$p = Re^{\phi\sqrt{-1}}, \quad q = Re^{-\phi\sqrt{-1}};$$

also $$\tan \phi = -\frac{K}{G}, \text{ and } R = \tfrac{1}{2}(G^2 + K^2)^{\frac{1}{2}} = (-H)^{\frac{3}{2}};$$

and finally, since $\omega = \cos \dfrac{2\pi}{3} \pm \sqrt{-1} \sin \dfrac{2\pi}{3} = e^{\pm\frac{2\pi}{3}\sqrt{-1}}$,

the three roots of the cubic equation

$$z^3 + 3Hz + G = 0,$$

viz. $$\sqrt[3]{p} + \sqrt[3]{q}, \quad \omega\sqrt[3]{p} + \omega^2\sqrt[3]{q}, \quad \omega^2\sqrt[3]{p} + \omega\sqrt[3]{q},$$

become

$$2(-H)^{\frac{1}{2}} \cos\frac{\phi}{3}, \quad -2(-H)^{\frac{1}{2}} \cos\frac{\pi \pm \phi}{3};$$

from which formulas we obtain the numerical values of the roots

of the cubic by aid of a table of sines and cosines. This process is not convenient in practice; and in general, for purposes of arithmetical calculation of real roots, the methods of solution of numerical equations to be hereafter explained (Chap. X.) should be employed.

58. **Expression of the Cubic as the Difference of two Cubes.**—Let the given cubic

$$ax^3 + 3bx^2 + 3cx + d = \phi(x)$$

be put under the form

$$z^3 + 3Hz + G,$$

where $z = ax + b$.

Now, assuming

$$z^3 + 3Hz + G = \frac{1}{\mu - \nu}\{\mu(z+\nu)^3 - \nu(z+\mu)^3\}, \qquad (1)$$

where μ and ν are quantities to be determined, the second side of this identity becomes, when reduced,

$$z^3 - 3\mu\nu z - \mu\nu(\mu + \nu).$$

Comparing coefficients,

$$\mu\nu = -H, \quad \mu\nu(\mu + \nu) = -G;$$

therefore

$$\mu + \nu = \frac{G}{H}, \quad \mu - \nu = \frac{a\sqrt{\Delta}}{H};$$

where $a^2\Delta = G^2 + 4H^3$, as in Art. 42;

also $$(z + \mu)(z + \nu) = z^2 + \frac{G}{H}z - H. \qquad (2)$$

Whence, putting for z its value, $ax + b$, we have from (1)

$$a^3\phi(x) = \left(\frac{G + a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G - a\Delta^{\frac{1}{2}}}{2H}\right)^3 - \left(\frac{G - a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G + a\Delta^{\frac{1}{2}}}{2H}\right)^3,$$

which is the required expression for $\phi(x)$ as the difference of two cubes.

By the aid of the identity just proved the cubic can be

resolved into its simple factors, and the solution of the equation completed. We proceed to obtain expressions for the roots of the equation $\phi(x) = 0$ in terms of μ and ν. Solving as a binomial cubic the equation

$$(\mu - \nu)\, a^2 \phi(x) \equiv \mu (z + \nu)^3 - \nu (z + \mu)^3 = 0,$$

we find the three following values for $z \equiv ax + b$:—

$$\sqrt[3]{\mu}\,\sqrt[3]{\nu}\,(\sqrt[3]{\mu} + \sqrt[3]{\nu}),$$

$$\sqrt[3]{\mu}\,\sqrt[3]{\nu}\,(\omega\sqrt[3]{\mu} + \omega^2\sqrt[3]{\nu}),$$

$$\sqrt[3]{\mu}\,\sqrt[3]{\nu}\,(\omega^2\sqrt[3]{\mu} + \omega\sqrt[3]{\nu}).$$

If now $\sqrt[3]{\mu}$ and $\sqrt[3]{\nu}$ be replaced by any pair of cube roots selected one from each of the two series

$$\sqrt[3]{\mu}, \quad \omega\sqrt[3]{\mu}, \quad \omega^2\sqrt[3]{\mu},$$

$$\sqrt[3]{\nu}, \quad \omega\sqrt[3]{\nu}, \quad \omega^2\sqrt[3]{\nu},$$

it will be seen that we shall get the same three values of z, the *order* only of these values changing according to the cube roots selected. It follows that the expression

$$\sqrt[3]{\mu}\,\sqrt[3]{\nu}\,(\sqrt[3]{\mu} + \sqrt[3]{\nu})$$

has three, and only three, values when the cube roots therein are taken in all generality. This form therefore is, in addition to that obtained in the last Article, a form proper to represent a root of a cubic equation (see (1), Art. 55).

The function (2) given above, when transformed and reduced, becomes, as may be easily seen,

$$\frac{a^2}{H}\{(ac - b^2)\,x^2 + (ad - bc)\,x + (bd - c^2)\}.$$

This quadratic, therefore, contains as factors the two binomials $ax + b + \mu$, $ax + b + \nu$, which occur in the above expression of $\phi(x)$ as the difference of two cubes.

59. **Solution of the Cubic by Symmetric Functions of the Roots.**—Since the three values of the expression

$$\tfrac{1}{3}\{\alpha+\beta+\gamma+\theta(\alpha+\omega\beta+\omega^2\gamma)+\theta^2(\alpha+\omega^2\beta+\omega\gamma)\},$$

when θ takes the values 1, ω, ω^2, are α, β, γ, it is plain that if the functions

$$\theta(\alpha+\omega\beta+\omega^2\gamma),\quad \theta^2(\alpha+\omega^2\beta+\omega\gamma)$$

were expressed in terms of the coefficients of the cubic, we could, by substituting their values in the formula given above, arrive at an algebraical solution of the cubic equation. Now this cannot be done directly by solving a quadratic equation; for, although the product of the two functions above written is a rational symmetric function of α, β, γ, their sum is not so. It will be found, however, that the sum of the cubes of the two functions in question is a symmetric function of the roots, and can, therefore, be expresssd by the coefficients, as we proceed to show. For convenience we adopt the notation

$$L\equiv\alpha+\omega\beta+\omega^2\gamma,\quad M\equiv\alpha+\omega^2\beta+\omega\gamma.$$

We have then

$$(\theta L)^3=A+B\omega+C\omega^2,\quad (\theta^2M)^3=A+B\omega^2+C\omega,$$

where

$$A=\alpha^3+\beta^3+\gamma^3+6\alpha\beta\gamma,\ B=3(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha),\ C=3(\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2);$$

from which we obtain

$$L^3+M^3=2\Sigma\alpha^3-3\Sigma\alpha^2\beta+12\alpha\beta\gamma=-27\frac{G}{a^3}\cdot$$

(Cf. Ex. 5, p. 44; Ex. 15, p. 50.)

Again,

$$(\theta L)(\theta^2M)=LM=\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta=-9\frac{H}{a^2};$$

whence $\qquad (\alpha+\omega\beta+\omega^2\gamma)^3,\quad (\alpha+\omega^2\beta+\omega\gamma)^3$

are the roots of the quadratic equation

$$t^2+3^3\frac{G}{a^3}t-3^6\frac{H^3}{a^6}=0.$$

Denoting the roots of this equation, viz.

$$\frac{3^3}{2a^3}\left(-G\pm\sqrt{G^2+4H^3}\right)$$

by t_1 and t_2, the original formula expressed in terms of the coefficients of the cubic gives for the three roots

$$\alpha = -\frac{b}{a} + \frac{1}{3}\left(\sqrt[3]{t_1} + \sqrt[3]{t_2}\right),$$

$$\beta = -\frac{b}{a} + \frac{1}{3}\left(\omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2}\right),$$

$$\gamma = -\frac{b}{a} + \frac{1}{3}\left(\omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2}\right).$$

It will be seen that the values of α, β, γ here arrived at are f the same form as those already obtained in Art. 56.

It is important to observe that the functions

$$(\alpha + \omega\beta + \omega^2\gamma)^3, \quad (\alpha + \omega^2\beta + \omega\gamma)^3$$

are remarkable as being the simplest functions of *three* quantities which have but *two* values when these quantities are interchanged in every way. It is owing to this property that the solution of a cubic equation can be reduced to that of a quadratic. Several functions of α, β, γ of this nature exist; and it will be proved in a subsequent chapter that any two such functions are connected by a rational linear relation in terms of the coefficients.

Having now completed the discussion of the different modes of algebraical solution of the cubic, we give some examples involving the principles contained in the preceding Articles.

Examples.

1. Resolve into simple factors the expression

$$(\beta-\gamma)^2(x-\alpha)^2 + (\gamma-\alpha)^2(x-\beta)^2 + (\alpha-\beta)^2(x-\gamma)^2.$$

Let $\quad U = (\beta-\gamma)(x-\alpha), \quad V = (\gamma-\alpha)(x-\beta), \quad W = (\alpha-\beta)(x-\gamma).$

Ans. $\frac{2}{3}(U + \omega V + \omega^2 W)(U + \omega^2 V + \omega W)$.

2. Prove that the several equations of the system

$$(\beta-\gamma)^3(x-\alpha)^3 = (\gamma-\alpha)^3(x-\beta)^3 = (\alpha-\beta)^3(x-\gamma)^3$$

have two factors common.

Making use of the notation in the last Example, we have

$$U^3 = V^3 = W^3;$$

whence

$$U^3 - V^3 = (U - V)(U^2 + UV + V^2) \equiv \tfrac{1}{2}(U - V)(U^2 + V^2 + W^2),$$

since

$$U + V + W \equiv 0;$$

therefore

$$(\beta - \gamma)^2 (x - \alpha)^2 + (\gamma - \alpha)^2 (x - \beta)^2 + (\alpha - \beta)^2 (x - \gamma)^2$$

is the common quadratic factor required.

3. Resolve into factors the expressions

$$(1).\quad (\beta - \gamma)^3 (x - \alpha)^3 + (\gamma - \alpha)^3 (x - \beta)^3 + (\alpha - \beta)^3 (x - \gamma)^3,$$

$$(2).\quad (\beta - \gamma)^5 (x - \alpha)^5 + (\gamma - \alpha)^5 (x - \beta)^5 + (\alpha - \beta)^5 (x - \gamma)^5,$$

$$(3).\quad (\beta - \gamma)^7 (x - \alpha)^7 + (\gamma - \alpha)^7 (x - \beta)^7 + (\alpha - \beta)^7 (x - \gamma)^7.$$

These factors can be written down at once from the results established in Ex. 40, p. 59. Using the notation of Ex. 1, and replacing $\alpha_1, \beta_1, \gamma_1$, in the example referred to, by U, V, W, we obtain the following :—

Ans. (1) $3UVW$; (2) $\frac{5}{2}(U^2 + V^2 + W^2)UVW$; (3) $\frac{7}{4}(U^2 + V^2 + W^2)^2 UVW$.

4. Express

$$(x - \alpha)(x - \beta)(x - \gamma)$$

as the difference of two cubes.

Assume

$$(x - \alpha)(x - \beta)(x - \gamma) = U_1^3 - V_1^3;$$

whence

$$U_1 - V_1 = \lambda (x - \alpha),$$

$$\omega U_1 - \omega^2 V_1 = \mu (x - \beta),$$

$$\omega^2 U_1 - \omega V_1 = \nu (x - \gamma).$$

Adding, we have

$$\lambda + \mu + \nu = 0, \quad \lambda\alpha + \mu\beta + \nu\gamma = 0;$$

and, therefore,

$$\lambda = \rho (\beta - \gamma), \quad \mu = \rho (\gamma - \alpha), \quad \nu = \rho (\alpha - \beta);$$

but $\lambda\mu\nu = 1$; whence

$$\frac{1}{\rho^3} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

Substituting these values of λ, μ, ν; and using the notation of Ex. 1,

$$U_1 - V_1 = \rho U, \quad \omega U_1 - \omega^2 V_1 = \rho V, \quad \omega^2 U_1 - \omega V_1 = \rho W;$$

whence

$$3U_1 = \rho (U + \omega^2 V + \omega W),$$

$$-3V_1 = \rho (U + \omega V + \omega^2 W);$$

and U_1 and V_1 are completely determined.

5. Prove that L and M are functions of the differences of the roots.

We have $\quad L = \alpha + \omega\beta + \omega^2\gamma = \alpha - h + \omega (\beta - h) + \omega^2 (\gamma - h)$

for all values of h, since $1 + \omega + \omega^2 = 0$; and giving to h the values α, β, γ, in succession, we obtain three forms for L in terms of the differences $\beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$. Similarly for M.

6. To express the product of the squares of the differences of the roots in terms of the coefficients.

We have

$$L+M=2\alpha-\beta-\gamma,\quad L+\omega^2M=(2\beta-\gamma-\alpha)\,\omega,\quad L+\omega M=(2\gamma-\alpha-\beta)\,\omega^2;$$

and, again,

$$L-M=(\beta-\gamma)(\omega-\omega^2),\quad \omega^2L-\omega M=(\gamma-\alpha)(\omega-\omega^2),\quad \omega L-\omega^2M=(\alpha-\beta)(\omega-\omega^2),$$

from which we obtain, as in Art. 26,

$$L^3+M^3=(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta),$$

$$L^3-M^3=-3\sqrt{-3}\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta);$$

and since

$$(L^3-M^3)^2\equiv(L^3+M^3)^2-4L^3M^3,$$

we have, substituting the values of L^3+M^3 and LM obtained in Art. 59,

$$a^6\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2=-27\,(G^2+4H^3).$$

(Cf. Art. 42.)

7. Prove the following identities:—

$$L^3+M^3\equiv\tfrac{1}{3}\{(2\alpha-\beta-\gamma)^3+(2\beta-\gamma-\alpha)^3+(2\gamma-\alpha-\beta)^3\},$$

$$L^3-M^3\equiv\sqrt{-3}\{(\beta-\gamma)^3+(\gamma-\alpha)^3+(\alpha-\beta)^3\}.$$

These are easily obtained by cubing and adding the values of

$$L+M,\ \&c.;\ L-M,\ \&c.,$$

in the preceding example.

8. To obtain expressions for L^2, M^2, &c., in terms of α, β, γ.

The following forms for L^2 and M^2 are obtained by subtracting

$(\alpha^2+\beta^2+\gamma^2)(1+\omega+\omega^2)\equiv 0$ from $(\alpha+\omega\beta+\omega^2\gamma)^2$, and $(\alpha+\omega^2\beta+\omega\gamma)^2$:—

$$-L^2=(\beta-\gamma)^2+\omega^2(\gamma-\alpha)^2+\omega\,(\alpha-\beta)^2,$$

$$-M^2=(\beta-\gamma)^2+\omega\,(\gamma-\alpha)^2+\omega^2(\alpha-\beta)^2.$$

In a similar manner, we find from these expressions

$$-L^4=(\beta-\gamma)^2\,(2\alpha-\beta-\gamma)^2+\omega\,(\gamma-\alpha)^2\,(2\beta-\gamma-\alpha)^2+\omega^2(\alpha-\beta)^2\,(2\gamma-\alpha-\beta)^2,$$

$$-M^4=(\beta-\gamma)^2\,(2\alpha-\beta-\gamma)^2+\omega^2(\gamma-\alpha)^2\,(2\beta-\gamma-\alpha)^2+\omega\,(\alpha-\beta)^2\,(2\gamma-\alpha-\beta)^2.$$

Also, without difficulty, we have the following forms for LM and L^2M^2:—

$$2LM=(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2,$$

$$L^2M^2=(\alpha-\beta)^2\,(\alpha-\gamma)^2+(\beta-\gamma)^2\,(\beta-\alpha)^2+(\gamma-\alpha)^2(\gamma-\beta)^2.$$

9. There are six functions of the type of L or M, viz.,

$$\alpha+\omega\beta+\omega^2\gamma,\quad \omega\alpha+\omega^2\beta+\gamma,\quad \omega^2\alpha+\beta+\omega\gamma,$$

$$\alpha+\omega^2\beta+\omega\gamma,\quad \omega\alpha+\beta+\omega^2\gamma,\quad \omega^2\alpha+\omega\beta+\gamma,$$

to form the equation whose roots are these six quantities.

These functions may be expressed as follows :—

$$L, \quad \omega L, \quad \omega^2 L,$$
$$M, \quad \omega M, \quad \omega^2 M;$$

hence they are the roots of the equation

$$(\phi - L)(\phi - \omega L)(\phi - \omega^2 L)(\phi - M)(\phi - \omega M)(\phi - \omega^2 M) = 0,$$

or

$$\phi^6 - (L^3 + M^3)\,\phi^3 + L^3 M^3 = 0.$$

Substituting for L and M from the equations

$$LM = -\frac{9H}{a^2}, \quad L^3 + M^3 = -27\frac{G}{a^3},$$

we have this equation expressed in terms of the coefficients as follows :—

$$\phi^6 + 3^3 \frac{G}{a^3}\phi^3 - 3^6 \frac{H^3}{a^6} = 0.$$

10. To form, in terms of L and M, the equation whose roots are the squares of the differences of the roots of the general cubic equation.

Let

$$\phi = (\alpha - \beta)^2;$$

hence, by former results,

$$\sqrt{-3\phi} = \omega L - \omega^2 M.$$

Rationalizing this, we obtain

$$\phi\,(\phi - LM)^2 + \frac{(L^3 - M^3)^2}{27} = 0,$$

which is the required equation.

In a similar manner, by the aid of the results of Ex. 8, the equation of squared differences of this equation, or the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2, \quad (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2, \quad (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2,$$

is obtained by substituting $-L^2$ and $-M^2$ for M and L, respectively, in the last equation; and this process may be repeated any number of times. Finally, all these equations may be easily expressed in terms of the coefficients of the cubic by means of the relations

$$LM = -9\frac{H}{a^2}, \quad \text{and} \quad L^3 + M^3 = -27\frac{G}{a^3}.$$

For instance, the first equation is

$$\phi\left(\phi + 9\frac{H}{a^2}\right)^2 + 27\,\frac{G^2 + 4H^3}{a^6} = 0.$$

(Cf. Art. 42.)

11. If α, β, γ and α', β', γ' be the roots of the cubic equations

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

$$a'x^3 + 3b'x^2 + 3c'x + d' = 0;$$

to form the equation which has for roots the six values of the function

$$\phi \equiv \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

The easiest mode of procedure is first to form the corresponding equation for the cubics deprived of their second terms, viz.,

$$z^3 + 3Hz + G = 0, \quad z^3 + 3H'z + G' = 0,$$

and thence deduce the equation in the general case; for in the case of the cubics so transformed the corresponding function

$$\begin{aligned}\phi_0 &\equiv (a\alpha + b)(a'\alpha' + b') + (a\beta + b)(a'\beta' + b') + (a\gamma + b)(a'\gamma' + b') \\ &\equiv aa'\phi - 3bb'.\end{aligned}$$

Substituting for the roots of the transformed equations their values expressed by radicals, we have

$$\begin{aligned}\phi_0 = (\sqrt[3]{p} + \sqrt[3]{q})(\sqrt[3]{p'} + \sqrt[3]{q'}) &+ (\omega\sqrt[3]{p} + \omega^2\sqrt[3]{q})(\omega\sqrt[3]{p'} + \omega^2\sqrt[3]{q'}) \\ &+ (\omega^2\sqrt[3]{p} + \omega\sqrt[3]{q})(\omega^2\sqrt[3]{p'} + \omega\sqrt[3]{q'}),\end{aligned}$$

which reduces to

$$\phi_0 = 3(\sqrt[3]{pq'} + \sqrt[3]{p'q}).$$

Cubing this, we find

$$\phi_0^3 - 27\sqrt[3]{pqp'q'}\,\phi_0 - 27(pq' + p'q) = 0.$$

Now, substituting for p and q, p' and q', their values given by the equations

$$x^2 + Gx - H^3 = 0, \quad x^2 + G'x - H'^3 = 0,$$

we have the six values of ϕ_0 given by the two cubic equations

$$\phi_0^3 - 27HH'\,\phi_0 - \frac{27}{2}(GG' \pm aa'\sqrt{\Delta\Delta'}) = 0,$$

where

$$a^2\Delta = G^2 + 4H^3, \quad \text{and} \quad a'^2\Delta' = G'^2 + 4H'^3.$$

Finally, substituting for ϕ_0 its value $aa'\phi - 3bb'$, and multiplying these cubics together, we have the required equation. It may be noticed that if one of the cubics be $x^3 - 1 = 0$, $\phi = \alpha + \omega\beta + \omega^2\gamma$, &c., which case has been already considered in Ex. 9. Mr. M. Roberts, *Dublin Exam. Papers*, 1855.

12. Form the equation whose roots are the several values of ρ, where

$$\rho = \frac{\alpha - \beta}{\beta - \gamma}.$$

Since

$$\alpha - (1 + \rho)\beta + \rho\gamma = 0;$$

substituting for α, β, γ their values in terms of p, q, and putting

$$\lambda = 1 - (1 + \rho)\omega + \rho\omega^2, \quad \mu = 1 - (1 + \rho)\omega^2 + \rho\omega,$$

we have

$$\lambda\sqrt[3]{p} + \mu\sqrt[3]{q} = 0.$$

Cubing, and substituting for p, q their values,

$$G(\lambda^3 + \mu^3) + a\sqrt{\Delta}(\lambda^3 - \mu^3) = 0.$$

Squaring,

$$a^2\Delta\lambda^3\mu^3 = H^3(\lambda^3 + \mu^3)^2,$$

and by previous results

$$\lambda\mu = 3(1 + \rho + \rho^2), \quad \lambda^3 + \mu^3 = -27\rho(1 + \rho);$$

substituting these values, we have the required equation

$$a^2\Delta(1 + \rho + \rho^2)^3 - 27H^3(\rho + \rho^2)^2 = 0.$$

13. Find the relation between the coefficients of the cubics

$$ax^3 + 3bx^2 + 3cx + d = 0,$$
$$a'x'^3 + 3b'x'^2 + 3c'x' + d' = 0,$$

when the roots are connected by the equation

$$\alpha(\beta' - \gamma') + \beta(\gamma' - \alpha') + \gamma(\alpha' - \beta') = 0.$$

Multiplying by $\omega - \omega^2$, this equation becomes

$$LM' = L'M.$$

Cubing, and introducing the coefficients, we find

$$G^2H'^3 = G'^2H^3,$$

the required relation.

14. Determine the condition in terms of the roots and coefficients that the cubics of Ex. 13 should become identical by the linear transformation

$$x' = px + q.$$

In this case

$$\alpha' = p\alpha + q, \quad \beta' = p\beta + q, \quad \gamma' = p\gamma + q.$$

Eliminating p and q, we have

$$\beta\gamma' - \beta'\gamma + \gamma\alpha' - \gamma'\alpha + \alpha\beta' - \alpha'\beta = 0,$$

which is the function of the roots considered in the last example. This relation, moreover, is unchanged if for α, β, γ; α', β', γ', we substitute

$$l\alpha + m, \quad l\beta + m, \quad l\gamma + m,$$
$$l'\alpha' + m', \quad l'\beta' + m', \quad l'\gamma' + m';$$

whence we may consider the cubics in the last example under the simple forms

$$z^3 + 3Hz + G = 0, \quad z'^3 + 3H'z' + G' = 0,$$

obtained by the linear transformations $z = ax + b$, $z' = a'x' + b'$; for if the condition holds for the roots of the former equations, it must hold for the roots of the latter. Now putting $z' = kz$, these equations become identical if

$$H' \equiv k^2H, \quad G' \equiv k^3G;$$

whence, eliminating k,

$$G^2H'^3 = G'^2H^3$$

is the required condition, the same as that obtained in Ex. 13. It may be observed that the reducing quadratics of the cubics necessarily become identical by the same transformation, viz.,

$$\frac{H'}{G'}(a'x' + b') = \frac{H}{G}(ax + b).$$

60. **Homographic Relation between two Roots of a Cubic.**—Before proceeding to the discussion of the biquadratic we prove the following important proposition relative to the cubic :—

The roots of the cubic are connected in pairs by a homographic relation in terms of the coefficients.

Referring to Exs. 13, 14, Art. 27, we have the relations

$$a_0^2\{\ (\beta-\gamma)^2 + \ (\gamma-\alpha)^2 + \ (\alpha-\beta)^2\} = 18(a_1^2 - a_0a_2),$$
$$a_0^2\{\alpha\ (\beta-\gamma)^2 + \beta\ (\gamma-\alpha)^2 + \gamma\ (\alpha-\beta)^2\} = \ 9(a_0a_3 - a_1a_2),$$
$$a_0^2\{\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2\} = 18(a_2^2 - a_1a_3).$$

Using the notation

$$a_0a_2 - a_1^2 \equiv H, \quad a_0a_3 - a_1a_2 \equiv 2H_1, \quad a_1a_3 - a_2^2 \equiv H_2;$$

multiplying the above equations by $\alpha\beta$, $-(\alpha+\beta)$, 1, respectively, and adding; since

$$\alpha^2 - \alpha(\alpha+\beta) + \alpha\beta \equiv 0, \quad \beta^2 - \beta(\alpha+\beta) + \alpha\beta \equiv 0,$$

we have

$$a_0^2(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)^2 = 18\{H\alpha\beta + H_1(\alpha+\beta) + H_2\};$$

but

$$a_0^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27\Delta \equiv 108(HH_2 - H_1^2)$$

(see Art. 42); whence

$$\pm\sqrt{-\frac{\Delta}{3}}\left(\frac{\alpha-\beta}{2}\right) = H\alpha\beta + H_1(\alpha+\beta) + H_2,$$

and, therefore,

$$H\alpha\beta + \left(H_1 + \frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\alpha + \left(H_1 - \frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\beta + H_2 = 0,$$

which is the required homographic relation. It is to be observed that the coefficients in this equation involve one irrational quantity, the second sign of which will give the relation between a different pair of the roots.

61. **First Solution by Radicals of the Biquadratic. Euler's Assumption.**—Let the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

be put under the form (Art. 37)

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where $z = ax + b$,

$$H \equiv ac - b^2, \quad I \equiv ae - 4bd + 3c^2, \quad G \equiv a^2d - 3abc + 2b^3.$$

To solve this equation (a biquadratic wanting the second term) Euler assumes as the general expression for a root

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

Squaring,

$$z^2 - p - q - r = 2(\sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}).$$

Squaring again, and reducing, we obtain the equation

$$z^4 - 2(p+q+r)z^2 - 8z\sqrt{p}\sqrt{q}\sqrt{r} + (p+q+r)^2 - 4(qr+rp+pq) = 0.$$

Comparing this equation with the former, we have

$$p+q+r = -3H, \quad qr+rp+pq = 3H^2 - \frac{a^2I}{4}, \quad \sqrt{p}\sqrt{q}\sqrt{r} = -\frac{G}{2};$$

and consequently p, q, r are the roots of the equation

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0; \tag{1}$$

or, since

$$-G^2 = 4H^3 - a^2HI + a^3J, \qquad \text{(Art. 37)},$$

where

$$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3,$$

this equation may be written in the form

$$4(t+H)^3 - a^2I(t+H) + a^3J = 0;$$

and finally, putting $t + H \equiv a^2\theta$, we obtain the equation

$$4a^3\theta^3 - Ia\theta + J = 0. \qquad (2)$$

This is called *the reducing cubic* of the biquadratic equation; and will in what follows be referred to by that name. When it is necessary to make a distinction between equations (1) and (2), we shall refer to the former as Euler's cubic.

Also, since $t \equiv b^2 - ac + a^2\theta$; if θ_1, θ_2, θ_3 be the roots of the reducing cubic, we have

$$p \equiv b^2 - ac + a^2\theta_1, \quad q \equiv b^2 - ac + a^2\theta_2, \quad r \equiv b^2 - ac + a^2\theta_3;$$

and, therefore,

$$z = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} + \sqrt{b^2 - ac + a^2\theta_3}.$$

If this formula be taken to represent a root of the biquadratic in z, it must be observed that the radicals involved have not complete generality; for if they had, eight values of z in place of four would be given by the formula. The proper limitation is imposed by the relation

$$\sqrt{p}\sqrt{q}\sqrt{r} = -\frac{G}{2},$$

which (lost sight of in squaring to obtain the value of pqr) requires such signs to be attached to each of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, that their product may maintain the sign determined by the above equation; thus—

$$\begin{aligned}\sqrt{p}\sqrt{q}\sqrt{r} = \sqrt{p}(-\sqrt{q})(-\sqrt{r}) &= (-\sqrt{p})\sqrt{q}(-\sqrt{r}) \\ &= (-\sqrt{p})(-\sqrt{q})\sqrt{r}\end{aligned}$$

are all the possible combinations of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ fulfilling this condition, provided that $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ retain the same signs throughout, whatever those signs may be. We may, however, remove all ambiguity as regards sign, and express in a single

algebraic formula the four values of z, by eliminating one of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ from the assumed value of z by means of the relation given above, and leaving the other two quantities unrestricted in sign. The expression for z becomes therefore

$$z = \sqrt{p} + \sqrt{q} - \frac{G}{2\sqrt{p}\sqrt{q}},$$

a formula free from all ambiguity, since it gives four, and only four, values of z when $\sqrt{p}$ and $\sqrt{q}$ receive their double signs: the sign given to each of these in the two first terms determining that which must be attached to it in the denominator of the third term. And finally, restoring to p, q, and z their values given before, we have

$$ax + b = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} - \frac{G}{2\sqrt{b^2 - ac + a^2\theta_1}\sqrt{b^2 - ac + a^2\theta_2}}$$

as *the complete algebraic solution of the biquadratic equation*; θ_1 and θ_2 being roots of the equation

$$4a^3\theta^3 - Ia\theta + J = 0.$$

To assist the student in justifying Euler's apparently arbitrary assumption as to the form of solution of the biquadratic, we remark that, the second term of the equation in z being absent, the sum of the four roots is zero, or $z_1 + z_2 + z_3 + z_4 = 0$; and consequently the functions $(z_1 + z_2)^2$, &c., of which there are in general *six* (the combinations of four quantities two and two), are in this case reduced to *three*; so that we may assume

$$(z_2 + z_3)^2 = (z_1 + z_4)^2 = 4p,$$

$$(z_3 + z_1)^2 = (z_2 + z_4)^2 = 4q,$$

$$(z_1 + z_2)^2 = (z_3 + z_4)^2 = 4r;$$

from which we have z_1, z_2, z_3, z_4, included in the formula

$$\sqrt{p} + \sqrt{q} + \sqrt{r}.$$

We now proceed to express the roots of Euler's cubic (1), and also those of the reducing cubic (2), in terms of the roots α, β, γ, δ of the given biquadratic in x. Attending to the remarks above made with reference to the signs of the radicals, we may write the four values of $z = ax + b$ as follows:—

$$\begin{aligned} a\alpha + b &= \sqrt{p} - \sqrt{q} - \sqrt{r}, \\ a\beta + b &= -\sqrt{p} + \sqrt{q} - \sqrt{r}, \\ a\gamma + b &= -\sqrt{p} - \sqrt{q} + \sqrt{r}, \\ a\delta + b &= \sqrt{p} + \sqrt{q} + \sqrt{r}; \end{aligned} \qquad (3)$$

from which may be immediately derived the following expressions for p, q, r the roots of Euler's cubic:—

$$\begin{aligned} p &= \frac{a^2}{16}(\beta + \gamma - \alpha - \delta)^2, \\ q &= \frac{a^2}{16}(\gamma + \alpha - \beta - \delta)^2, \\ r &= \frac{a^2}{16}(\alpha + \beta - \gamma - \delta)^2. \end{aligned} \qquad (4)$$

Subtracting in pairs the equations (3), and making use of the relations above written between p, q, r and θ_1, θ_2, θ_3, we easily establish the following useful relations connecting the differences of the roots of the cubics (1) and (2) with the differences of the roots of the biquadratic:—

$$\begin{aligned} 4(q - r) &= 4a^2(\theta_2 - \theta_3) = -a^2(\beta - \gamma)(\alpha - \delta), \\ 4(r - p) &= 4a^2(\theta_3 - \theta_1) = -a^2(\gamma - \alpha)(\beta - \delta), \\ 4(p - q) &= 4a^2(\theta_1 - \theta_2) = -a^2(\alpha - \beta)(\gamma - \delta). \end{aligned} \qquad (5)$$

Finally, from these equations, by aid of the relation $\theta_1 + \theta_2 + \theta_3 = 0$, we derive the values of θ_1, θ_2, θ_3 in terms of α, β, γ, δ, viz.,

$$\begin{aligned} 12\theta_1 &= (\gamma - \alpha)(\beta - \delta) - (\alpha - \beta)(\gamma - \delta), \\ 12\theta_2 &= (\alpha - \beta)(\gamma - \delta) - (\beta - \gamma)(\alpha - \delta), \\ 12\theta_3 &= (\beta - \gamma)(\alpha - \delta) - (\gamma - \alpha)(\beta - \delta). \end{aligned} \qquad (6)$$

Examples.

1. When the biquadratic has two equal roots, the reducing cubic has two equal roots, and conversely.

2. When the biquadratic has three roots equal, all the roots of the reducing cubic vanish, and consequently $I = 0$, $J = 0$.

3. When the biquadratic has two distinct pairs of equal roots, two of the roots of Euler's cubic vanish, and consequently $G = 0$, $a^2 I - 12H^2 = 0$.

4. Prove the following relations between the biquadratic and Euler's cubic with respect to the nature of the roots :—

(1). When the roots of the biquadratic are all real, the roots of Euler's cubic are all real and positive.

(2). When the roots of the biquadratic are all imaginary, the roots of Euler's cubic are all real, two being negative and one positive.

(3). When the biquadratic has two real and two imaginary roots, Euler's cubic has two imaginary roots and one real positive root.

These results follow readily from equations (1) when the proper forms are substituted for α, β, γ, δ in the values of p, q, r. It is to be observed that all possible cases are here comprised, the biquadratic being supposed not to have equal roots. It follows that the converse of each of these propositions is true. Hence, when Euler's cubic has all its roots real and positive, we may conclude that all the roots of the biquadratic are real; when Euler's cubic has negative roots, we conclude that all the roots of the biquadratic are imaginary; and when Euler's cubic has imaginary roots, we conclude that the biquadratic has two real and two imaginary roots.

5. Prove that the roots of the biquadratic and the roots of the reducing cubic are connected by the following relations :—

(1). When the roots of the biquadratic are either all real, or all imaginary, the roots of the reducing cubic are all real; and, conversely, when the roots of the reducing cubic are all real, the roots of the biquadratic are either all real or all imaginary.

(2). When the biquadratic has two real, and two imaginary roots, the reducing cubic has two imaginary roots; and, conversely, when the reducing cubic has two imaginary roots, the biquadratic has two real and two imaginary roots.

These results follow readily from the preceding example, since the roots of the two cubics (1) and (2) are connected by a real linear relation.

6. When H is positive, the biquadratic has imaginary roots.

For in that case the roots of Euler's cubic cannot be all positive.

7. When I is negative, the biquadratic has two real and two imaginary roots.

For the reducing cubic has in that case two imaginary roots (Ex. 12, p. 33).

8. When H and J are both positive, all the roots of the biquadratic are imaginary.

For, since J is positive, the reducing cubic has a real negative root; therefore also Euler's cubic has a real negative root, since $t = a^2\theta - H$, and H is positive; and this is case (2) of Ex. 4. It is implied in this proof that the leading coefficient a is positive; if aJ be substituted for J in the statement of the proposition no restriction as to the sign of a is necessary.

9. Show that the two biquadratic equations

$$A_0x^4 + 6A_2x^2 \pm 4A_3x + A_4 = 0$$

have the same reducing cubic.

10. Find the reducing cubic of the two biquadratic equations

$$x^4 - 6lx^2 \pm 8x\sqrt{l^3 + m^3 + n^3 - 3lmn} + 3(4mn - l^2) = 0.$$

Ans. $\theta^3 - 3mn\theta - (m^3 + n^3) = 0$.

11. Prove that the eight roots of the equation

$$\{x^4 - 6lx^2 + 3(4mn - l^2)\}^2 = 64(l^3 + m^3 + n^3 - 3lmn)x^2$$

are given by the formula

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}.$$

(Compare Ex. 20, p. 34.)

12. If the expression

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}$$

be a root of the equation

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

determine H, I, J in terms of l, m, n.

Ans. $H = -l$, $I = 12mn$, $J = -4(m^3 + n^3)$.

13. Write down the formulas which express the root of a biquadratic in the particular cases when $I = 0$, and $J = 0$.

14. Express, by the aid of the reducing cubic, I and J in terms of the differences of the roots α, β, γ, δ. (See Exs. 16, 18, Art. 27.)

15. Express the product of the squares of the differences of the roots α, β, γ, δ in terms of I and J.

By means of the equations (5) above given, and the equation (2), p. 82, we obtain the result as follows:—

$$a^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2(\alpha - \delta)^2(\beta - \delta)^2(\gamma - \delta)^2 = 256(I^3 - 27J^2).$$

16. What is the quantity under the *final* square root (viz., that which occurs under the cube root in the solution of the reducing cubic) in the formula expressing a root? *Ans.* $27J^2 - I^3$.

17. Prove that the coefficients of the equation of squared differences of the biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ may be expressed in terms a_0, H, I, and J.

Removing the second term from the equation, we obtain

$$y^4 + \frac{6H}{a_0^2} y^2 + \frac{4G}{a_0^3} y + \frac{a_0^2 I - 3H^2}{a_0^4} = 0;$$

and changing the signs of the roots, we have

$$y^4 + \frac{6H}{a_0^2} y^2 - \frac{4G}{a_0^3} y + \frac{a_0^2 I - 3H^2}{a_0^4} = 0.$$

These transformations leave the functions $(\alpha - \beta)^2$, &c., unaltered; but G becomes $-G$, the other coefficients of the latter equation remaining unchanged; therefore G can enter the coefficients of the equation of squared differences in *even* powers only. And by aid of the identity of Art. 37, G^2 may be eliminated, introducing a_0, H, I, J. In a similar manner we may prove that every even function of the differences of the roots α, β, γ, δ may be expressed in terms of a_0, H, I, J, the function G of odd degree not entering.

62. **Second Solution by Radicals of the Biquadratic.**—Let the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

be put, as before, under the form

$$z^4 + 6Hz^2 + 4Gz + a^2 I - 3H^2 = 0,$$

where $z = ax + b$.

We now assume as the general expression for a root of this equation

$$z = \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q},$$

a formula involving three independent radicals, $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$.

Squaring twice, and reducing, we have

$$(z^2 - qr - rp - pq)^2 = 4pqr(2z + p + q + r),$$

or

$$z^4 - 2(qr + rp + pq)z^2 - 8pqrz + (qr + rp + pq)^2 - 4(p + q + r)pqr = 0.$$

Comparing this equation with the former equation in z, we easily find

$$qr + rp + pq = -3H, \quad pqr = -\frac{G}{2}, \quad p + q + r = \frac{a^2 I - 12H^2}{2G};$$

whence, p, q, r are the roots of the equation

$$2Gt^3 + (12H^2 - a^2 I)t^2 - 6HGt + G^2 = 0.$$

This equation may be readily transformed into Euler's cubic, or making directly the substitution

$$t = \frac{\frac{1}{2}G}{H - a^2\theta},$$

and putting for G^2 its value in terms of H, I, and J, we may reduce it to the standard form of the reducing cubic, viz.,

$$4a^3\theta^3 - Ia\theta + J = 0.$$

It is important to observe that in the present method of solution we meet with no ambiguity corresponding to that of Art. 61; for the expression here assumed as the value of z has, in virtue of the double signs of the radicals contained in it, *only four values*, while the form assumed for z in the preceding Article has eight values. This appears from the identical equation

$$2(\sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}) \equiv (\sqrt{p} + \sqrt{q} + \sqrt{r})^2 - p - q - r,$$

which shows that the number of distinct values of the radical expression of the present Article is the same as the number of values of $(\sqrt{p} + \sqrt{q} + \sqrt{r})^2$, namely four.

In order to express p, q, r in terms of the roots α, β, γ, δ of the biquadratic, we have, giving to x the four values α, β, γ, δ,

$$\begin{aligned}
z_1 &\equiv a\alpha + b = \sqrt{q}\sqrt{r} - \sqrt{r}\sqrt{p} - \sqrt{p}\sqrt{q},\\
z_2 &\equiv a\beta + b = -\sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} - \sqrt{p}\sqrt{q},\\
z_3 &\equiv a\gamma + b = -\sqrt{q}\sqrt{r} - \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q},\\
z_4 &\equiv a\delta + b = \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}.
\end{aligned}$$

The student may easily satisfy himself that no combination of the signs of the radicals can lead to any value different from these four.

From the values of $z_2 + z_3 - z_1 - z_4$, and $z_2z_3 - z_1z_4$, we obtain

$$a(\beta + \gamma - \alpha - \delta) = -4\sqrt{q}\sqrt{r},$$

$$a^2(\beta\gamma - \alpha\delta) + ab(\beta + \gamma - \alpha - \delta) = 4p\sqrt{q}\sqrt{r}.$$

From these and similar equations we have, employing the relation $G = -2pqr$, the following modes of expressing p, q, r in terms of the roots $\alpha, \beta, \gamma, \delta$:—

$$-p = a\frac{\beta\gamma - \alpha\delta}{\beta + \gamma - \alpha - \delta} + b = \frac{8G}{a^2(\beta + \gamma - \alpha - \delta)^2},$$

$$-q = a\frac{\gamma\alpha - \beta\delta}{\gamma + \alpha - \beta - \delta} + b = \frac{8G}{a^2(\gamma + \alpha - \beta - \delta)^2},$$

$$-r = a\frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta} + b = \frac{8G}{a^2(\alpha + \beta - \gamma - \delta)^2}.$$

63. **Resolution of the Quartic into its Quadratic Factors.**—Let the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

be supposed to be expressed as the difference of two squares* in the form

$$(ax^2 + 2bx + c + 2a\theta)^2 - (2Mx + N)^2.$$

Multiplying the given quartic by a, and comparing it with this expression, we have the following equations to determine M, N, and θ:—

$$M^2 = b^2 - ac + a^2\theta, \quad MN = bc - ad + 2ab\theta, \quad N^2 = (c + 2a\theta)^2 - ae.$$

Eliminating M and N from these equations, we find

$$4a^3\theta^3 - (ae - 4bd + 3c^2)\,a\theta + ace + 2bcd - ad^2 - eb^2 - c^3 = 0,$$

which is the reducing cubic before obtained.

From this equation we have three values of θ (θ_1, θ_2, θ_3), with three corresponding values of M^2, MN, N^2; and thus all the coefficients of the assumed form for the quartic are deter-

* The reduction of the quartic to the difference of two squares was the method first employed for the solution of the equation of the fourth degree. This mode of solution is due to *Ferrari*, although by some writers ascribed to *Simpson* (see note A). The method explained in the following Article, in which the quartic is equated directly to the product of two quadratic factors, is due to *Descartes*.

mined in three distinct ways; moreover, it should be noticed that to each value of M corresponds a *single* value of N, since

$$MN = bc - ad + 2ab\theta.$$

The quartic

$$(ax^2 + 2bx + c + 2a\theta)^2 - (2Mx + N)^2$$

may plainly be resolved into the two quadratic factors

$$ax^2 + 2(b - M)x + c + 2a\theta - N,$$

$$ax^2 + 2(b + M)x + c + 2a\theta + N.$$

When θ receives the three values θ_1, θ_2, θ_3, we obtain the three pairs of quadratic factors of the original quartic, and the problem is completely solved.

In order to make clear the connexion between the present solution and the solution by radicals, let us suppose that the roots of the quadratic factors in the order above written are β, γ and α, δ; and that the roots of the remaining pairs of quadratic factors are similarly γ, α and β, δ; α, β and γ, δ. We have, therefore,

$$\beta + \gamma = -\frac{2}{a}(b - M_1), \quad \gamma + \alpha = -\frac{2}{a}(b - M_2), \quad \alpha + \beta = -\frac{2}{a}(b - M_3),$$

$$\alpha + \delta = -\frac{2}{a}(b + M_1), \quad \beta + \delta = -\frac{2}{a}(b + M_2), \quad \gamma + \delta = -\frac{2}{a}(b + M_3),$$

where

$$M_1 \equiv \sqrt{b^2 - ac + a^2\theta_1}, \quad M_2 \equiv \sqrt{b^2 - ac + a^2\theta_2}, \quad M_3 \equiv \sqrt{b^2 - ac + a^2\theta_3}.$$

Subtracting the last equations in pairs, we find

$$\beta + \gamma - \alpha - \delta = 4\frac{M_1}{a}, \quad \gamma + \alpha - \beta - \delta = 4\frac{M_2}{a}, \quad \alpha + \beta - \gamma - \delta = 4\frac{M_3}{a};$$

and since

$$\alpha + \beta + \gamma + \delta = -4\frac{b}{a},$$

we obtain

$$a\alpha + b = -M_1 + M_2 + M_3,$$

$$a\beta + b = M_1 - M_2 + M_3,$$

$$a\gamma + b = M_1 + M_2 - M_3,$$

$$a\delta + b = -M_1 - M_2 - M_3.$$

It appears, therefore, that the roots of the biquadratic are here expressed separately by formulas analogous to those of Art. 61. The values of M^2, viz. M_1^2, M_2^2, M_3^2, are in fact identical with the roots of Euler's cubic in the preceding Article. There exists also with regard to the signs of the radicals involved in M_1, M_2, M_3 a restriction similar to that of Art. 61; since, in virtue of the assumptions above made with respect to the roots of the quadratic factors, we have the equation

$$a^3(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta) = 64 M_1 M_2 M_3,$$

which implies the following relation (see Ex. 20, p. 52):—

$$M_1 M_2 M_3 = \tfrac{1}{2} G;$$

and by means of this relation the signs of M_1, M_2, M_3 are restricted in the manner explained in the previous Article.

By aid of the equation last written we can eliminate M_3 from the expressions for the roots, and thus obtain, as in Art. 61, all the roots of the biquadratic in a *single* formula, viz.,

$$ax + b = M_1 + M_2 - \frac{G}{2M_1 M_2},$$

in which the radicals $M_1 \equiv \sqrt{b^2 - ac + a^2\theta_1}$, and $M_2 \equiv \sqrt{b^2 - ac + a^2\theta_2}$ are taken in complete generality.

Examples.

1. Form the equation whose roots are λ, μ, ν, viz.,

$$\beta\gamma + \alpha\delta, \quad \gamma\alpha + \beta\delta, \quad \alpha\beta + \gamma\delta.$$

Adding the last coefficients of the quadratic factors of the quartic, we have

$$\beta\gamma + \alpha\delta = 4\theta_1 + 2\frac{c}{a},$$

$$\gamma\alpha + \beta\delta = 4\theta_2 + 2\frac{c}{a},$$

$$\alpha\beta + \gamma\delta = 4\theta_3 + 2\frac{c}{a},$$

where θ_1, θ_2, θ_3 are the roots of the reducing cubic; hence the required equation.

Ans. $(ax - 2c)^3 - 4I(ax - 2c) + 16J = 0.$

(Cf. Exs. 4, 5, Art. 39.)

2. Express, by means of the equations of the preceding example, the roots of the reducing cubic in terms of the roots of the biquadratic.

Substituting for $\frac{2c}{a}$ its value in terms of α, β, γ, δ, we find immediately

$$12\theta_1 = 2\lambda - \mu - \nu \equiv (\gamma - \alpha)(\beta - \delta) - (\alpha - \beta)(\gamma - \delta),$$

$$12\theta_2 = 2\mu - \nu - \lambda \equiv (\alpha - \beta)(\gamma - \delta) - (\beta - \gamma)(\alpha - \delta),$$

$$12\theta_3 = 2\nu - \lambda - \mu \equiv (\beta - \gamma)(\alpha - \delta) - (\gamma - \alpha)(\beta - \delta).$$

(Cf. (6), Art. 61.)

3. Verify, by means of the expressions for θ_1, θ_2, θ_3 in Ex. 1, the conclusions of Ex. 5, Art. 61, with respect to the manner in which the roots of the biquadratic and reducing cubic are related.

4. Form the equation whose roots are the functions

$$\tfrac{1}{8}(\beta\gamma - \alpha\delta)(\beta + \gamma - \alpha - \delta), \quad \tfrac{1}{8}(\gamma\alpha - \beta\delta)(\gamma + \alpha - \beta - \delta), \quad \tfrac{1}{8}(\alpha\beta - \gamma\delta)(\alpha + \beta - \gamma - \delta).$$

From the quadratic factors of the quartic we find

$$\frac{4M_1}{a} = \beta + \gamma - \alpha - \delta, \quad -\frac{2N_1}{a} = \beta\gamma - \alpha\delta;$$

also

$$M_1N_1 = bc - ad + 2ab\,\theta_1 = -a^2\phi_1,$$

the roots of the required cubic being represented by ϕ_1, ϕ_2, ϕ_3.

We obtain, therefore, the required equation by a linear transformation of the reducing cubic.

Ans. $(a^2\phi + bc - ad)^3 - b^2I(a^2\phi + bc - ad) - 2b^3J = 0.$

5. Form the equation whose roots are

$$\frac{\beta\gamma - \alpha\delta}{\beta + \gamma - \alpha - \delta}, \quad \frac{\gamma\alpha - \beta\delta}{\gamma + \alpha - \beta - \delta}, \quad \frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta}.$$

If ϕ denote any one of these functions indifferently, and θ the corresponding root of the reducing cubic, we have, employing former results,

$$-2\phi = \frac{MN}{M^2} = \frac{bc - ad + 2ab\theta}{b^2 - ac + a^2\theta};$$

and thus we obtain the required equation by a homographic transformation of the reducing cubic. This formula may be put under the more convenient form

$$a\phi + b = \frac{\frac{1}{2}G}{a^2\theta - H},$$

by means of which we obtain the required cubic in the following form:—

$$2G(a\phi + b)^3 + (a^2I - 12H^2)(a\phi + b)^2 - 6HG(a\phi + b) - G^2 = 0,$$

which, expanded and divided by a^3, becomes

$$2G\phi^3 + (a^2e + 6b^2c - 9ac^2 + 2abd)\phi^2 + 2(abe + 2b^2d - 3acd)\phi + b^2e - ad^2 = 0.$$

(Cf. Ex. 14, p. 88.)

6. Form the equation whose roots are

$$\frac{a^2}{4}(\beta\gamma - \alpha\delta)^2, \quad \frac{a^2}{4}(\gamma\alpha - \beta\delta)^2, \quad \frac{a^2}{4}(\alpha\beta - \gamma\delta)^2.$$

These are the three values of N^2 in the foregoing Article. Representing, as before, one of these values by ϕ, we find that the required equation may be obtained from the reducing cubic by means of the homographic transformation

$$\phi = \frac{2bcd - ad^2 - eb^2 + 4abd\theta}{c - a\theta}.$$

7. Form the equation whose roots are

$$\frac{\beta\gamma - \alpha\delta}{(\beta + \gamma)\,\alpha\delta - (\alpha + \delta)\,\beta\gamma}, \quad \frac{\gamma\alpha - \beta\delta}{(\gamma + \alpha)\,\beta\delta - (\beta + \delta)\,\gamma\alpha}, \quad \frac{\alpha\beta - \gamma\delta}{(\alpha + \beta)\,\gamma\delta - (\gamma + \delta)\,\alpha\beta}.$$

The required equation is obtained from the reducing cubic by the homographic transformation

$$2\phi = \frac{cd - be + 2ad\theta}{d^2 - ce + ae\theta}.$$

This result may be derived from Ex. 5 by changing the roots into their reciprocals, and making the corresponding changes in the coefficients.

64. **The Resolution of the Quartic into Quadratic Factors. Second Method**—Let the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

be supposed to be resolved into the quadratic factors

$$a(x^2 + 2px + q)(x^2 + 2p'x + q').$$

We have, by comparing these two forms, the equations

$$p + p' = 2\frac{b}{a}, \quad q + q' + 4pp' = 6\frac{c}{a}, \quad pq' + p'q = 2\frac{d}{a}, \quad qq' = \frac{e}{a}. \qquad (1)$$

If now we had any fifth equation of the form

$$F(p, q, p', q') = \phi,$$

we could eliminate p, p', q, q'; and thus find an equation giving the several values of ϕ.

The fifth equation might be assumed to be $pp' = \phi$, or $q + q' = \phi$; and in each case ϕ would be determined by a cubic equation, since each of these functions, when expressed in terms of the

roots of the biquadratic, has three values only. It is more convenient, however, to assume

$$\phi = \frac{c}{a} - pp' = \frac{1}{4}\left(q + q' - \frac{2c}{a}\right),$$

the two functions of p, p', q, q' here involved being equal by the second of equations (1). We easily find, by the aid of those equations,

$$pq + p'q' = \frac{4abc - 2a^2d}{a^3} + \frac{8b\phi}{a};$$

and eliminating p, p', q, q', by means of the identical relation

$$(p^2 + p'^2)(q^2 + q'^2) \equiv (pq' - p'q)^2 + (pq + p'q')^2,$$

there results the equation

$$4a^3\phi^3 - Ia\phi + J = 0,$$

which is the reducing cubic obtained by the previous methods of solution.

Having thus found pp', or $q + q'$, we may complete the resolution of the quartic by means of the equations (1).

The reason for the assumption above made with regard to the form of the fifth equation is obvious. From a comparison of the assumed values of ϕ with the equations of Ex. 1, Art. 63, it appears that ϕ is the same as θ in the preceding Article; and therefore we foresee that the elimination of p, p', q, q', must lead to an equation in ϕ identical with the reducing cubic before obtained. In general, if ϕ represent any function of the differences of λ, μ, ν, and consequently an *even* function of the differences of α, β, γ, δ (see Ex. 18, Art. 27), the equation whose roots are the different values of ϕ cannot involve any functions of the coefficients except a, H, I, and J.

If ϕ be assumed equal to any of the expressions in the second of the following examples, the equation in ϕ whose roots are the different values of this expression is formed as in the above instance by the elimination of p, p', q, q'.

EXAMPLES.

1. Resolve into quadratic factors

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2.$$

Comparing this form with the product

$$(z^2 + 2pz + q)(z^2 - 2pz + q'),$$

we find the following equation for p :—

$$4p^6 + 12\,Hp^4 + 12\left(H^2 - \frac{a^2I}{12}\right)p^2 - G^2 = 0; \qquad \text{(cf. (1), Art. 61)}$$

and putting

$$a^2\phi = p^2 + H \equiv \tfrac{1}{4}(q + q' - 2H),$$

this equation, when divided by a^3, becomes

$$4a^3\phi^3 - Ia\phi + J = 0.$$

2. If a quartic be resolved into the two quadratic factors

$$x^2 + px + q, \quad x^2 + p'x + q',$$

prove that ϕ is determined by a cubic equation when it has all possible values corresponding to each of the following types:—

$$q + q', \quad \frac{q - q'}{p - p'}, \quad \frac{pq' - p'q}{p - p'}, \quad \frac{pq' - p'q}{q - q'}.$$

$$(p - p')^2, \ (p - p')(q - q'), \ (q - q')^2, \ (pq' - p'q)^2;$$

and by an equation of the sixth degree when it has all values corresponding to

$$p, q, \quad p - p', \quad q - q', \quad pq' - p'q, \quad \text{or} \quad p^2 - 4q.$$

Expressing these functions in terms of the roots, the number of possible values of each function becomes apparent.

65. Transformation of the Biquadratic into the Reciprocal Form.—To effect this transformation we make the linear substitution $x = ky + \rho$ in the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

which then assumes the form

$$ak^4y^4 + 4U_1k^3y^3 + 6U_2k^2y^2 + 4U_3ky + U_4 = 0,$$

where

$$U_1 \equiv a\rho + b, \ \ U_2 \equiv a\rho^2 + 2b\rho + c, \ \ U_3 \equiv a\rho^3 + 3b\rho^2 + 3c\rho + d, \ \ \&c.$$

(See Art. 35.) If this equation be reciprocal, we have two equations to determine k and ρ, viz.,

$$ak^4 = U_4, \ \ k^3U_1 = kU_3;$$

eliminating k, we have the following equation for ρ :—

$$aU_3^2 - U_1^2 U_4 = 0;$$

and since

$$k^2 = \frac{U_3}{U_1} = \frac{a\rho^3 + 3b\rho^2 + 3c\rho + d}{a\rho + b},$$

there are two values of k, equal with opposite signs, corresponding to each value of ρ.

The equation

$$aU_3^2 - U_1^2 U_4 = 0,$$

when reduced by the substitutions (Arts. 36, 37)

$$a^2 U_3 \equiv U_1^3 + 3HU_1 + G,$$

$$a^3 U_4 \equiv U_1^4 + 6HU_1^2 + 4GU_1 + a^2 I - 3H^2,$$

becomes

$$2GU_1^3 + (a^2 I - 12H^2)\,U_1^2 - 6GHU_1 - G^2 = 0, \qquad (1)$$

which is a cubic equation determining $U_1 \equiv a\rho + b$; and if we put

$$a\rho + b = \frac{\frac{1}{2}G}{a^2\theta - H},$$

θ is determined by the standard reducing cubic

$$4a^3\theta^3 - Ia\theta + J = 0.$$

This transformation* may be employed to solve the biquadratic; and it is important to observe that the cubic (1) which here presents itself differs from the cubic of Art. 62 only in having roots with contrary signs.

We proceed now to express k and ρ in terms of $\alpha, \beta, \gamma, \delta$, the roots of the biquadratic equation. Since the equation in y, obtained by putting $x = ky + \rho$, is reciprocal, its roots are of the form $y_1, y_2, \frac{1}{y_2}, \frac{1}{y_1}$; hence we may write

$$\alpha = ky_1 + \rho, \quad \beta = ky_2 + \rho, \quad \gamma = k\frac{1}{y_2} + \rho, \quad \delta = k\frac{1}{y_1} + \rho;$$

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* This method of solving the biquadratic by transforming it to the reciprocal form was given by Mr. S. S. Greatheed in the *Camb. Math. Journ.*, vol. i.

and, therefore,

$$(\alpha-\rho)(\delta-\rho) = (\beta-\rho)(\gamma-\rho) = k^2,$$

from which we find

$$\rho = \frac{\beta\gamma - \alpha\delta}{\beta+\gamma-\alpha-\delta},$$

and
$$-k^2 = \frac{(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}{(\beta+\gamma-\alpha-\delta)^2}.$$

An important geometrical interpretation may be given to the quantities k and ρ which enter into this transformation. Let the distances OA, OB, OC, OD, of four points A, B, C, D, on a right line from a fixed origin O on the line be determined by the roots α, β, γ, δ, of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0;$$

also let O_1, O_2, O_3 be the centres; and F_1, F_1'; F_2, F_2'; F_3, F_3', the foci of the three systems of involution determined by the three following pairs of quadratics:—

$$(x-\beta)(x-\gamma) = 0, \quad (x-\alpha)(x-\delta) = 0;$$
$$(x-\gamma)(x-\alpha) = 0, \quad (x-\beta)(x-\delta) = 0;$$
$$(x-\alpha)(x-\beta) = 0, \quad (x-\gamma)(x-\delta) = 0.$$

We have then the equations

$$O_1B \,.\, O_1C = O_1A \,.\, O_1D = O_1F_1^2, \text{ \&c.},$$

which, transformed and compared with the equations

$$(\beta-\rho)(\gamma-\rho) = (\alpha-\rho)(\delta-\rho) = k^2, \text{ \&c.},$$

prove that the three values of ρ are OO_1, OO_2, OO_3, the distances of the three centres of involution from the fixed origin O. Also since $O_1F_1^2 = k^2$, k has six values represented geometrically by the distances

$$O_1F_1,\ O_1F_1';\ O_2F_2,\ O_2F_2';\ O_3F_3,\ O_3F_3',$$

where $O_1F_1 + O_1F_1' = 0$, &c., as the distances are measured in opposite directions.

We can from geometrical considerations alone find the positions of the centres and foci of involution in terms of $\alpha, \beta, \gamma, \delta$, and thus confirm the results just established, as follows:—

Since the systems $\{F_1BF_1'C\}$ and $\{F_1AF_1'D\}$ are harmonic,

$$\frac{2}{F_1F_1'} = \frac{1}{F_1B} + \frac{1}{F_1C} = \frac{1}{F_1A} + \frac{1}{F_1D};$$

and if x represent the distance of F_1 or F_1' from the fixed origin O, we have

$$\frac{1}{x-\beta} + \frac{1}{x-\gamma} = \frac{1}{x-\alpha} + \frac{1}{x-\delta}.$$

Solving this equation, we find

$$x = \frac{\beta\gamma - \alpha\delta}{\beta+\gamma-\alpha-\delta} \pm \frac{\sqrt{-(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}}{\beta+\gamma-\alpha-\delta},$$

or $\quad x = \rho \pm k$,

whence $\quad \rho = \dfrac{OF_1 + OF_1'}{2}, \quad k = \pm\dfrac{OF_1 - OF_1'}{2} = \pm\, O_1F_1.$

Example.

Transform the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

to the reciprocal form.

The assumption $x = ky + \rho$ leads to the equation

$$-GU_1^3 + 3H^2U_1^2 + H^3 = 0, \text{ where } U_1 \equiv a\rho + b.$$

The values of ρ are easily seen to be

$$\frac{\beta\gamma - \alpha^2}{\beta+\gamma-2\alpha}, \quad \frac{\gamma\alpha - \beta^2}{\gamma+\alpha-2\beta}, \quad \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}.$$

The geometrical interpretation in this case is, that if three points A', B', C' be taken on the axis such that A' is the harmonic conjugate of A with respect to B and C, B' of B with respect to C and A, and C' of C with respect to A and B; then we have the following values of ρ and k:—

$$\rho = \frac{OA + OA'}{2}, \quad k = \frac{OA - OA'}{2}.$$

For the values of OA', OB', OC', in terms of α, β, γ, see Ex. 13, p. 88.

66. **Solution of the Biquadratic by Symmetric Functions of the Roots.**—The possibility of reducing the solution of the biquadratic to that of a cubic by the present method depends on the possibility of forming functions of the four roots α, β, γ, δ, which admit of only three values when these roots are interchanged in every way. It will be seen on referring to Ex. 2, Art. 64, that several functions of this nature exist. These, like the analogous functions of Art. 59, possess an important property to be proved hereafter, viz., any two such sets of three are so related that any one function of either set is connected with some one function of the other set by a rational homographic relation in terms of the coefficients.

For the purposes of the present solution we employ the functions already referred to in Art. 55, since they lead in the most direct manner to the expressions for the roots of the biquadratic in terms of the coefficients. We proceed accordingly to form the equation whose roots are the three values of

$$t = \left(\frac{\alpha + \theta\beta + \theta^2\gamma + \theta^3\delta}{4}\right)^2,$$

when the roots are interchanged in every way, and $\theta = -1$.

These values are

$$t_1 = \left(\frac{\beta + \gamma - \alpha - \delta}{4}\right)^2, \quad t_2 = \left(\frac{\gamma + \alpha - \beta - \delta}{4}\right)^2, \quad t_3 = \left(\frac{\alpha + \beta - \gamma - \delta}{4}\right)^2;$$

and since

$$(\beta + \gamma - \alpha - \delta)^2 = \Sigma\alpha^2 + 2\lambda - 2\mu - 2\nu,$$

$$\Sigma(\alpha - \beta)^2 = 3\Sigma\alpha^2 - 2\lambda - 2\mu - 2\nu = -48\frac{H}{a^2},$$

we find the following values of t_1, t_2, t_3:—

$$\frac{2\lambda - \mu - \nu}{12} - \frac{H}{a^2}, \quad \frac{2\mu - \nu - \lambda}{12} - \frac{H}{a^2}, \quad \frac{2\nu - \lambda - \mu}{12} - \frac{H}{a^2};$$

whence
$$t_1 + t_2 + t_3 = -3\frac{H}{a^2}.$$

Again, since

$$\Sigma(2\mu-\nu-\lambda)(2\nu-\lambda-\mu)=-3(\lambda^2+\mu^2+\nu^2-\mu\nu-\nu\lambda-\lambda\mu)=-\frac{3}{2}\Sigma(\mu-\nu)^2,$$

and

$$\Sigma(\mu-\nu)^2 = 24\frac{I}{a^2},$$

we have

$$t_2t_3 + t_3t_1 + t_1t_2 = 3\frac{H^2}{a^4} - \frac{1}{96}\Sigma(\mu-\nu)^2 = \frac{3H^2}{a^4} - \frac{I}{4a^2};$$

also

$$t_1 t_2 t_3 = \frac{G^2}{4a^6}.$$

Hence the equation whose roots are t_1, t_2, t_3 becomes

$$(a^2t)^3 + 3H(a^2t)^2 + \left(3H^2 - \frac{a^2I}{4}\right)(a^2t) - \frac{G^2}{4} = 0;$$

or, substituting for G^2 its value from Art. 37,

$$4(a^2t + H)^3 - a^2I(a^2t + H) + a^3J = 0,$$

which is transformed into the standard reducing cubic by the substitution $a^2t + H = a^2\theta$.

To determine α, β, γ, δ we have the following equations :—

$$-\alpha+\beta+\gamma-\delta = 4\sqrt{t_1},\quad \alpha-\beta+\gamma-\delta = 4\sqrt{t_2},\quad \alpha+\beta-\gamma-\delta = 4\sqrt{t_3},$$

along with

$$\alpha+\beta+\gamma+\delta = -4\frac{b}{a};$$

from which we find

$$\alpha = -\frac{b}{a} - \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3},$$

$$\beta = -\frac{b}{a} + \sqrt{t_1} - \sqrt{t_2} + \sqrt{t_3},$$

$$\gamma = -\frac{b}{a} + \sqrt{t_1} + \sqrt{t_2} - \sqrt{t_3},$$

$$\delta = -\frac{b}{a} - \sqrt{t_1} - \sqrt{t_2} - \sqrt{t_3}.$$

We have also from the above values of $\sqrt{t_1}$, $\sqrt{t_2}$, $\sqrt{t_3}$ the equation

$$\sqrt{t_1}\sqrt{t_2}\sqrt{t_3} = \frac{G}{2a^3},$$

by means of which one radical can be expressed in terms of the other two, and the general formula for a root shown to be the same as those previously given.

It is convenient, in connexion with the subject of this Article, to give some account of two functions of the roots of the biquadratic which possess properties analogous to those established in Art. 59 for corresponding functions of the roots of a cubic. Adopting a notation similar to that of the Article referred to, we may write these functions in terms of λ, μ, ν in the following form:—

$$L \equiv (\beta\gamma + a\delta) + \omega\,(\gamma a + \beta\delta) + \omega^2(a\beta + \gamma\delta),$$

$$M \equiv (\beta\gamma + a\delta) + \omega^2(\gamma a + \beta\delta) + \omega\,(a\beta + \gamma\delta).$$

By means of the equations of Ex. 1, Art. 63, these functions can be expressed in terms of the roots of the reducing cubic in the form

$$\tfrac{1}{4}L = \theta_1 + \omega\theta_2 + \omega^2\theta_3, \quad \tfrac{1}{4}M = \theta_1 + \omega^2\theta_2 + \omega\theta_3.$$

They may also be expressed, by aid of the equation of the present Article connecting t and θ, in terms of the values of t_1, t_2, t_3, as follows:—

$$\tfrac{1}{4}L = t_1 + \omega t_2 + \omega^2 t_3, \quad \tfrac{1}{4}M = t_1 + \omega^2 t_2 + \omega t_3.$$

The functions L and M are as important in the theory of the biquadratic as the functions of Art. 59 in the theory of the cubic. The cubes of these expressions are the simplest functions of four quantities which have but *two* values when these quantities are interchanged in every way; they are the roots of the reducing quadratic of the reducing cubic above written, and underlie every solution of the biquadratic which has been given.

Examples.

1. Show that L and M are functions of the differences of $\alpha, \beta, \gamma, \delta$.

Increasing $\alpha, \beta, \gamma, \delta$ by h, L and M remain unaltered, since $1 + \omega + \omega^2 = 0$.

2. To find in terms of the coefficients the product of the squares of the differences of the roots $\alpha, \beta, \gamma, \delta$.

From the values of L and M in terms of $\theta_1, \theta_2, \theta_3$, we find easily

$$12\theta_1 = L + M, \qquad L - M = (\beta-\gamma)(\alpha-\delta)(\omega^2-\omega),$$

$$12\theta_2 = \omega^2 L + \omega M, \qquad \omega^2 L - \omega M = (\gamma-\alpha)(\beta-\delta)(\omega^2-\omega),$$

$$12\theta_3 = \omega L + \omega^2 M, \qquad \omega L - \omega^2 M = (\alpha-\beta)(\gamma-\delta)(\omega^2-\omega).$$

Again, from these equations, multiplying the terms on both sides together, and remembering that $\theta_1, \theta_2, \theta_3$ are the roots of

$$4a^3\theta^3 - Ia\theta + J = 0,$$

we find

$$L^3 + M^3 = -432\frac{J}{a^3},$$

$$L^3 - M^3 = 3\sqrt{-3}\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta);$$

also, adding the squares of the same terms, we have

$$2LM = 24\frac{I}{a^2} = (\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2;$$

and, since

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4L^3M^3,$$

substituting for these quantities their values derived from former equations, we have finally

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2 = 256(I^3 - 27J^2).$$

3. Show by a comparison of the equations of Art. 59 with those of the present Article that the results of the former may be extended to the biquadratic by changing

$$\beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta \text{ into } -(\beta-\gamma)(\alpha-\delta),\ -(\gamma-\alpha)(\beta-\delta),\ -(\alpha-\beta)(\gamma-\delta),$$

respectively; and, at the same time, H into $-\frac{4}{3}I$, and G into $16J$.

67. Equation of Squared Differences of a Biquadratic.—In a previous chapter (Art. 44) an account was given of the general problem of the formation of the equation of differences. It was proposed by Lagrange to employ this equation in practice for the purpose of separating the roots of a given numerical equation; and with a view to such application

he calculated the general forms of the equation of squared differences in the cases of equations of the fourth and fifth degrees wanting the second term (see *Traité de la Résolution des Equations Numériques*, 3rd ed., ch. v., and note III.). Although for practical purposes the methods of separation of the roots to be hereafter explained are to be preferred; yet, in connexion with the subjects of the present chapter, the equation of squared differences of the biquadratic is of sufficient interest to be given here. We proceed accordingly to calculate this equation for a biquadratic written in the most general form. It will appear, in accordance with what was proved in Ex. 17, Art. 61, that the coefficients of the resulting equation can all be expressed in terms of a, H, I, and J.

The problem is equivalent to expressing the following product in terms of the coefficients of the biquadratic

$$\{\phi-(\beta-\gamma)^2\}\{\phi-(\gamma-\alpha)^2\}\{\phi-(\alpha-\beta)^2\}\{\phi-(\alpha-\delta)^2\}\{\phi-(\beta-\delta)^2\}\{\phi-(\gamma-\delta)^2\}.$$

The most convenient mode of procedure is to group these six factors in pairs, and to express the three products (which we denote by Π_1, Π_2, Π_3) separately in terms of the roots of the reducing cubic, and finally to express the product $\Pi_1\Pi_2\Pi_3$ in terms of a, H, I, J.

$$\Pi_1 = \phi^2 - \{(\beta-\gamma)^2 + (\alpha-\delta)^2\}\,\phi + (\beta-\gamma)^2(\alpha-\delta)^2;$$

and, by aid of the results of Art. 61 we easily derive the following expressions for $(\beta-\gamma)^2$, $(\alpha-\delta)^2$:—

$$4\left(\sqrt{\theta_2-\frac{H}{a^2}}-\sqrt{\theta_3-\frac{H}{a^2}}\right)^2, \qquad 4\left(\sqrt{\theta_2-\frac{H}{a^2}}+\sqrt{\theta_3-\frac{H}{a^2}}\right)^2;$$

hence, without difficulty,

$$\Pi_1 = \phi^2 + \left(8\theta_1 + 16\frac{H}{a^2}\right)\phi + 4\frac{I}{a^2} - 48\theta_2\theta_3.$$

Introducing now for brevity the notation

$$16H = a^2P, \quad 4I = a^2Q, \quad 16J = a^3R, \quad \phi^2 + P\phi + Q = \Psi,$$

Π_1 becomes $\Psi + 8\theta_1\phi - 48\theta_2\theta_3$.

Reducing the product $\Pi_1\Pi_2\Pi_3$ by the result of Example 18, page 89, we obtain

$$\Psi^3 + 3Q\Psi^2 - (4Q\phi^2 + 18R\phi)\Psi - (8R\phi^3 + 12Q^2\phi^2 + 36QR\phi + 27R^2) = 0.$$

Finally, restoring the value of Ψ, we have the equation of squared differences expressed in terms of P, Q, R, as follows:—

$$\phi^6 + 3P\phi^5 + (3P^2 + 2Q)\phi^4 + (P^3 + 8PQ - 26R)\phi^3$$
$$+ (6P^2Q - 7Q^2 - 18PR)\phi^2 + 9Q(PQ - 6R)\phi + 4Q^3 - 27R^2 = 0.$$

The following is the final equation in terms of a, H, I, J* :—

$$a^6\phi^6 + 48a^4H\phi^5 + 8a^2(96H^2 + a^2I)\phi^4 + 32(128H^3 + 16a^2HI - 13a^3J)\phi^3$$
$$+ 16(384H^2I - 7a^2I^2 - 288aHJ)\phi^2 + 1152(2HI - 3aJ)I\phi + 256(I^3 - 27J^2) = 0.$$

It should be observed that the value above obtained for Π_1 can be expressed as a quadratic function of θ_1 by aid of the equation $\theta_2\theta_3 = \theta_1^2 - \frac{I}{4a^2}$, and the subsequent calculation might have been conducted by eliminating θ_1 between this quadratic and the reducing cubic.

68. **Criterion of the Nature of the Roots of the Biquadratic.**—Before proceeding with this investigation it is necessary to repeat what was before stated (Art. 43), that when any condition with respect to the nature of the roots of an algebraic equation is expressed by the sign of a function of the coefficients, these coefficients are supposed to represent real numerical quantities. It is assumed also, as in the Article referred to, that the leading coefficient does not vanish.

Using as before Δ to represent that function of the coefficients (called the *discriminant*) which is, when multiplied by a positive numerical factor, equal to the product of the squares of the differences of the roots, we have, from the results established in preceding Articles, the equation

$$a_0^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2 = 256\Delta,$$

where $$\Delta \equiv I^3 - 27J^2.$$

It will be found convenient in what follows to arrange the discussion of the nature of the roots under three heads, according as—(1) Δ *vanishes*, or (2) *is negative*, or (3) *is positive.*

(1) *When Δ vanishes, the equation has equal roots.* This is evident from the value of Δ above written. Four distinct cases may be noticed—(a) *when two roots only are equal*, in which case I and J do not vanish separately; (β) *when three roots are equal*, in which case $I = 0$, and $J = 0$, separately (see Ex. 2, Art. 61); (γ) *when*

* The equation of squared differences was first given in this form by Mr. M. Roberts in the *Nouvelles Annales de Mathématiques*, vol. xvi.

two distinct pairs of roots are equal, in which case we have the conditions $G = 0$, $a^2I - 12H^2 = 0$ (Ex. 3, Art. 61). It can be readily proved by means of the identity of Art. 37 that these conditions imply the equation $\Delta = 0$; hence these two equations, along with the equation $\Delta = 0$, are equivalent to two independent conditions only. Finally, we may have—(e) *all the roots equal;* in which case may be derived from Art. 61 the three independent conditions $H = 0$, $I = 0$, and $J = 0$. These may be written in a form analogous to the corresponding conditions in case (4) of Art. 43.

(2) *When Δ is negative, the equation has two real and two imaginary roots.*—This follows from the value of Δ in terms of the roots: for when all the roots are real Δ is plainly positive; and when the proper imaginary forms, viz., $h \pm k\sqrt{-1}$, $h' \pm k'\sqrt{-1}$, are substituted for $\alpha, \beta, \gamma, \delta$, it readily appears that Δ is positive also when all the roots are imaginary.

(3) *When Δ is positive, the roots of the equation are either all real or all imaginary.*—This follows also from the value of Δ, for we can show by substituting for α, β the forms $h \pm k\sqrt{-1}$ that Δ is negative when two roots are real and two imaginary. In the case, therefore, when Δ is positive, this function of the coefficients is not by itself sufficient to determine completely the nature of the roots, for it remains still doubtful whether the roots are all real or all imaginary. The further conditions necessary to discriminate between these two cases may, however, be obtained from Euler's cubic (Art. 61) as follows:—In order that the roots of this cubic should be all real and positive, it is necessary that the signs should be alternately positive and negative; and when the signs are of this nature the cubic cannot have a real negative root. We can, therefore, derive, by the aid of Ex. 4, Art. 61, the following general conclusion applicable to this case:—*When Δ is positive the roots of the biquadratic are all imaginary in every case except when the following conditions are fulfilled, viz., H negative, and $a^2I - 12H^2$ negative; in which case the roots are all real.*

Examples.

1. Show that if H be positive, or if $H = 0$ (and G not $= 0$), the cubic will have a pair of imaginary roots.

2. Show that if H be negative, the cubic will have its roots—(1) all real and unequal, (2) two equal, or (3) two imaginary, according as G^2 is—(1) less than, (2) equal to, or (3) greater than $-4H^3$.

3. If the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

have two roots equal to α; prove

$$-\alpha = \frac{H_2}{H_1} = \frac{H_1}{H},$$

where $\quad a_0a_2 - a_1^2 \equiv H, \quad a_0a_3 - a_1a_2 \equiv 2H_1, \quad a_1a_3 - a_2^2 \equiv H_2.$

4. If $$ax^3 + 3bx^2 + 3cx + d + k(x - r)^3$$

be a perfect cube, prove

$$(ac - b^2)r^2 + (ad - bc)r + (bd - c^2) = 0.$$

5. Find the condition that the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

may be capable of being written under the form

$$l(x - \alpha_1)^3 + m(x - \beta_1)^3 + n(x - \gamma_1)^3,$$

where α_1, β_1, γ_1 are the roots of the cubic

$$a_1x^3 + 3b_1x^2 + 3c_1x + d_1 = 0.$$

Comparing the forms we have

$$a = l + m + n,$$
$$-b = l\alpha_1 + m\beta_1 + n\gamma_1,$$
$$c = l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2,$$
$$-d = l\alpha_1^3 + m\beta_1^3 + n\gamma_1^3.$$

Also $$a_1\alpha_1^3 + 3b_1\alpha_1^2 + 3c_1\alpha_1 + d_1 = 0, \text{ \&c.}$$

Whence, multiplying these equations by d_1, $3c_1$, $3b_1$, a_1, respectively, and adding, we find the required condition

$$(ad_1 - a_1d) - 3(bc_1 - b_1c) = 0.$$

6. If α, β, γ be the roots of the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0;$$

rationalize the equation

$$\sqrt[4]{x - \alpha} + \sqrt[4]{x - \beta} + \sqrt[4]{x - \gamma} = 0;$$

and express the result in terms of the coefficients a_0, a_1, a_2, a_3.

Ans. $125U_1^4 + 360HU_1^2 + 128GU_1 - 48H^2 = 0.$

7. If α_1, β_1, and α_2, β_2 be the roots of the quadratic equations

$$a_1x^2 + 2b_1x + c_1 = 0, \quad a_2x^2 + 2b_2x + c_2 = 0;$$

find the equation whose roots are the four values of $\alpha_1\alpha_2$.

Let $H_1 \equiv a_1c_1 - b_1^2$, $H_2 \equiv a_2c_2 - b_2^2$.

Ans. $(a_1a_2\phi^2 - 2b_1b_2\phi + c_1c_2)^2 - 4H_1H_2\phi^2 = 0.$

N.B.—This and the two following Examples may be solved by expressing ϕ by radicals involving the coefficients.

8. Employing the notation of Ex. 7, form the equation whose roots are the four values of $\dfrac{\alpha_1 + \alpha_2}{2}$.

Let $2K_{12} \equiv a_1c_2 + a_2c_1 + 2b_1b_2$.

Ans. $(2a_1a_2\phi^2 + 2(a_1b_2 + a_2b_1)\phi + K_{12})^2 - H_1H_2 = 0.$

In this Example the resulting biquadratic is such that $G = 0$.

9. In the same case, if $\phi = \frac{1}{2}(\alpha_1 - \alpha_2)^2$, form the equation whose roots are the several values of ϕ.

Let $M \equiv a_1b_2 - a_2b_1$, $2H_{12} \equiv a_1c_2 + a_2c_1 - 2b_1b_2$.

Ans. $\{(a_1a_2\phi + H_{12})^2 - 2M^2\phi + H_1H_2\}^2 = 4H_1H_2(a_1a_2\phi + H_{12})^2.$

10. Show that when the biquadratic has a double root, the cubic whose roots are the values of ρ (Art. 65) has the *same* double root; and find what this cubic becomes when the biquadratic has three roots equal.

11. If H and J be both positive, prove directly (without the aid of Euler's cubic) that the roots of the biquadratic are all imaginary.

It appears from the expression for H in terms of the roots (Ex. 19, p. 52) that when H is positive there must be at least one pair of imaginary roots $h \pm k\sqrt{-1}$. Now diminishing all the roots by h, and dividing them by k (which transformations will not alter the character of the other pair of roots γ, δ, nor the signs of H and J), the biquadratic may be put under the form

$$(x^2 + 4px + q)(x^2 + 1),$$

or $$x^4 + 4px^3 + 6cx^2 + 4px + q, \text{ where } 6c = q + 1;$$

whence $$H = c - p^2, \quad I = q - 4p^2 + 3c^2,$$

$$J = qc + 2p^2c - p^2(q + 1) - c^3 = c(q - 4p^2 - c^2),$$

and therefore

$$q - 4p^2 = c^2 + \frac{J}{c} = (H + p^2)^2 + \frac{J}{H + p^2},$$

or $$-\left(\frac{\gamma - \delta}{2k}\right)^2 = \left(H + p^2\right)^2 + \frac{J}{H + p^2},$$

proving that γ and δ are imaginary when H and J are positive (cf. Ex. 8, Art. 61).

12. If the biquadratic has two distinct pairs of equal roots, prove directly the relations

$$a_0^2 I = 12H^2, \quad a_0^3 J = 8H^3.$$

In this case the biquadratic divided by a_0 assumes the form

$$(x-\alpha)^2(x-\beta)^2 \equiv \left\{\left(x - \frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2\right\}^2 = \left(\frac{z^2-k^2}{a_0^2}\right)^2,$$

where $$z = a_0 x + a_1, \quad \text{and} \quad \frac{k}{a_0} = \frac{\alpha-\beta}{2};$$

whence, comparing the forms

$$z^4 - 2k^2z^2 + k^4$$

and $$z^4 + 6Hz^2 + 4Gz + a_0^2 I - 3H^2,$$

we find $$3H = -k^2, \quad G = 0, \quad a_0^2 I - 3H^2 = k^4,$$

from which the above relations immediately follow. The student will easily establish the identity of these relations with those of Ex. 3, Art. 61. Also it should be noticed that in this case only one square root is involved in the solution of the biquadratic (coming from the solution of the quadratic $(x-\alpha)(x-\beta)$).

13. Find the condition that the biquadratic may be capable of being put under the form

$$l(x^2 + 2px + q)^2 + m(x^2 + 2px + q) + n.$$

In this case the second and fourth coefficients are removed by the same transformation, and the general solution involves only two square roots.

Ans. $G = 0$.

14. Prove that J vanishes for the biquadratic

$$m(x-n)^4 - n(x-m)^4.$$

15. If the roots of a biquadratic, $\alpha, \beta, \gamma, \delta$ represent the distances of four points from an origin on a right line; prove that when these points form a harmonic division on the line the roots of Euler's cubic are in arithmetic progression, and the roots of the cubic of Art. 62 in harmonic progression.

16. Form the equation whose roots are the six anharmonic functions of four points in a right line determined by the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

The six anharmonic ratios are

$$\phi_1, \frac{1}{\phi_1}, \quad \phi_2, \frac{1}{\phi_2}, \quad \phi_3, \frac{1}{\phi_3},$$

where
$$\phi_1 = -\frac{(\alpha-\beta)(\gamma-\delta)}{(\gamma-\alpha)(\beta-\delta)} = \frac{\lambda-\mu}{\lambda-\nu} = \frac{\theta_1-\theta_2}{\theta_1-\theta_3},$$
$$\phi_2 = -\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\beta)(\gamma-\delta)} = \frac{\mu-\nu}{\mu-\lambda} = \frac{\theta_2-\theta_3}{\theta_2-\theta_1},$$
$$\phi_3 = -\frac{(\gamma-\alpha)(\beta-\delta)}{(\beta-\gamma)(\alpha-\delta)} = \frac{\nu-\lambda}{\nu-\mu} = \frac{\theta_3-\theta_1}{\theta_3-\theta_2};$$
also the equation whose roots are
$$(\beta-\gamma)(\alpha-\delta), \quad (\gamma-\alpha)(\beta-\delta), \quad (\alpha-\beta)(\gamma-\delta)$$
is one of the cubics
$$a_0^3t^3 - 12a_0It \pm 16\sqrt{I^3 - 27J^2} = 0.$$

The equation whose roots are the ratios, with sign changed, of the roots of *either* of these cubics is
$$4\Delta(\phi^2-\phi+1)^3 - 27I^3\phi^2(\phi-1)^2 = 0 \qquad \text{(see Ex. 15, p. 88)},$$
where
$$\Delta \equiv I^3 - 27J^2.$$

The roots of the equation in ϕ are the six anharmonic ratios. This equation can be written in a more expressive form, as will appear from the following propositions :—

(*a*). The six anharmonic ratios may be expressed in terms of any one of them, as follows :—
$$\phi, \ \frac{1}{\phi}, \quad 1-\phi, \ \frac{1}{1-\phi}, \quad \frac{\phi-1}{\phi}, \ \frac{\phi}{\phi-1}.$$

From the identical equation
$$(\beta-\gamma)(\alpha-\delta) + (\gamma-\alpha)(\beta-\delta) + (\alpha-\beta)(\gamma-\delta) \equiv 0$$
we have the relations
$$\phi_1 + \frac{1}{\phi_3} = 1, \quad \phi_2 + \frac{1}{\phi_1} = 1, \quad \phi_3 + \frac{1}{\phi_2} = 1,$$
which determine all the anharmonic ratios in terms of any one of them.

(*b*). If two of the anharmonic ratios become equal, the six values of ϕ are $-\omega$ and $-\omega^2$, each occurring three times; and in this case $I = 0$.

For suppose $\phi_1 = \phi_2$; we have then from the second of the above relations
$$\phi_1^2 - \phi_1 + 1 = 0,$$
whence
$$\phi_1 = -\omega, \text{ or } -\omega^2;$$
and substituting either of these values for ϕ in (*a*), we find all the anharmonic ratios.

Also, since
$$\frac{\lambda-\mu}{\lambda-\nu} + \frac{\mu-\nu}{\lambda-\mu} = 0, \text{ or } \Sigma(\mu-\nu)^2 = 0,$$
we have
$$I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0.$$

(*c*). When one of the ratios is harmonic, the six values of ϕ are -1, 2, $\frac{1}{2}$, each occurring twice; and in this case $J = 0$; for if

$$\phi_1 = -1, \ \frac{\lambda - \mu}{\lambda - \nu} = -1, \text{ or } 2\lambda - \mu - \nu = 0,$$

one of the factors of J (see Ex. 18, p. 52).

(*d*). These results, as well as the converse propositions, may be proved by writing the sextic in ϕ under the following form:—

$$4I^3 \{(\phi + 1)(\phi - 2)(\phi - \tfrac{1}{2})\}^2 = 27J^2\{(\phi + \omega)(\phi + \omega^2)\}^3.$$

17. Solve the equation

$$\left(\frac{x^2 + 14x + 1}{\rho^4 + 14\rho^2 + 1}\right)^3 = \frac{x(x-1)^4}{\rho^2(\rho^2 - 1)^4}.$$

Ans. $\rho^2, \ \frac{1}{\rho^2}, \ \left(\frac{1 + \theta\sqrt{\rho}}{1 - \theta\sqrt{\rho}}\right)^4$, where $\theta^4 = 1$.

18. Express $\Sigma(\alpha - \beta)^4(\gamma - \delta)^2$ as a rational function of $\theta_1, \theta_2, \theta_3$; and ultimately in terms of the coefficients of the quartic.

Ans. $-128\,\Sigma(\theta_2 - \theta_3)^2\left(\theta_1 + \frac{2H}{a^2}\right) = -\frac{96}{a^4}(4HI + 3aJ)$.

19. Express

$$(\beta^2 - \gamma^2)^2(\alpha^2 - \delta^2)^2 + (\gamma^2 - \alpha^2)^2(\beta^2 - \delta^2)^2 + (\alpha^2 - \beta^2)^2(\gamma^2 - \delta^2)^2$$

as a rational function of $\theta_1, \theta_2, \theta_3$.

This symmetric function is equivalent to

$$(\mu^2 - \nu^2)^2 + (\nu^2 - \lambda^2)^2 + (\lambda^2 - \mu^2)^2 = 256\Sigma(\theta_2 - \theta_3)^2\left(\theta_1 - \frac{c}{a}\right)^2.$$

20. Form the equation whose roots are the several products in pairs of the roots of a biquadratic.

The required equation is the product of three factors of the type

$$(\phi - \beta\gamma)(\phi - \alpha\delta) = \phi^2 - \lambda\phi + \frac{e}{a} = \phi^2 - 2\frac{c}{a}\phi + \frac{e}{a} - 4\phi\theta_1.$$

Ans. $(a\phi^2 - 2c\phi + e)^3 - 4I\phi^2(a\phi^2 - 2c\phi + e) + 16J\phi^3 = 0$.

21. Form the equation whose roots are the several values of $\frac{\alpha + \beta}{2}$, where $\alpha, \beta, \gamma, \delta$ are the roots of a biquadratic.

The required equation is the product of three factors of the type

$$\left(\phi - \frac{\beta + \gamma}{2}\right)\left(\phi - \frac{\alpha + \delta}{2}\right) = \phi^2 + 2\frac{b}{a}\phi + \frac{\mu + \nu}{4} = \phi^2 + 2\frac{b}{a}\phi + \frac{c}{a} - \theta_1.$$

Ans. $4(a\phi^2 + 2b\phi + c)^3 - I(a\phi^2 + 2b\phi + c) + J = 0$.

22. Prove

$$\Sigma \frac{1}{(\alpha-\beta)^2} = \frac{9I}{2}\left(\frac{3aJ - 2HI}{I^3 - 27J^2}\right).$$

From the expressions for α, β, γ, δ in terms of θ_1, θ_2, θ_3, we have

$$\Sigma \frac{1}{(\alpha-\beta)^2} = -\frac{1}{2a^2}\left\{\frac{a^2\theta_1 + 2H}{(\theta_2-\theta_3)^2} + \frac{a^2\theta_2 + 2H}{(\theta_3-\theta_1)^2} + \frac{a^2\theta_3 + 2H}{(\theta_1-\theta_2)^2}\right\},$$

which may be expressed in terms of a, H, I, J, as above.

23. Prove $$\Sigma \frac{\theta_1{}^m}{(\theta_2-\theta_3)^2} = 0,$$

if $I = 0$, and m of the form $3p$ or $3p+1$, p being a positive integer.

24. Prove that

$$U \equiv ax^2 + cy^2 + ez^2 + 2dyz + 2czx + 2bxy$$

can be resolved into the sum or difference of two squares if

$$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3 = 0.$$

Here $$aU \equiv (ax + by + cz)^2 + (ac - b^2)y^2 + 2(ad - bc)yz + (ae - c^2)z^2,$$

and $$(ac - b^2)y^2 + 2(ad - bc)yz + (ae - c^2)z^2$$

is a perfect square if

$$(ac - b^2)(ae - c^2) = (ad - bc)^2,$$

or $J = 0$.

25. If α, β, γ, δ be the roots of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

solve, in terms of the coefficients a_0, a_1, &c., the equation

$$\sqrt{x-\alpha} + \sqrt{x-\beta} + \sqrt{x-\gamma} + \sqrt{x-\delta} = 0.$$

When $$\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + \sqrt{\delta} = 0$$

is rationalized, and the coefficients substituted for α, β, γ, δ, we have

$$(3a_0a_2 - 2a_1^2)^2 = a_0^3a_4.$$

Now, substituting U_0, U_1, U_2, U_3, U_4 for a_0, a_1, a_2, a_3, a_4, and reducing, we find

$$a_0x + a_1 = \frac{1}{G}\left(3H^2 - \frac{a_0^2I}{4}\right).$$

26. To express the solution of the biquadratic in terms of a single root of the reducing cubic.

Substituting $x' + \rho$ for x in the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

we have

$$ax'^4 + 4U_1x'^3 + 6U_2x'^2 + 4U_3x' + U_4 = 0.$$

As there are here two independent variables at our disposal, it is allowable to make the assumptions

$$ax'^4 + 6U_2x'^2 + U_4 = 0, \quad U_1x'^2 + U_3 = 0.$$

Eliminating x'^2, and reducing as in Art. 65, we have

$$4U_2^3 - IU_2 + J = 0;$$

whence $U_2 = a\theta$, where θ is a root of the reducing cubic, and therefore

$$U_1 = a\rho + b = \sqrt{a^2\theta - H}.$$

Again,

$$x'^2 = -\frac{U_3}{U_1} = -\frac{1}{a^2}\left(U_1^2 + 3H + \frac{G}{U_1}\right);$$

whence, finally, since $x = x' + \rho$, or $ax + b = U_1 + ax'$, we have

$$ax + b = \sqrt{a^2\theta - H} + \sqrt{-a^2\theta - 2H - \frac{G}{\sqrt{a^2\theta - H}}},$$

an expression which has only four values.

This expression might of course be obtained from the resulting formula of Art. 61, or from that of Art. 63. The method of arriving at it in the present Example is a distinct method of solving the biquadratic.

27. Prove that every rational algebraic function of a root θ of a given cubic equation can in general be reduced to the form

$$\frac{C_0 + C_1\theta}{D_0 + D_1\theta}.$$

Let the given function be $\dfrac{\phi(\theta)}{\psi(\theta)}$, where $\phi(\theta)$ and $\psi(\theta)$ are rational integral functions of θ of any order. By successive substitutions from the given cubic each of these may be reduced to a quadratic. Hence the given function is reducible to the form

$$\frac{c_0 + c_1\theta + c_2\theta^2}{d_0 + d_1\theta + d_2\theta^2}.$$

Equating this to the form written above, and reducing by the given cubic, we obtain an identical equation, viz.

$$L_0 + L_1\theta + L_2\theta^2 \equiv 0,$$

where L_0, L_1, L_2 are linear functions of C_0, C_1, D_0, D_1, We have, therefore, the three equations $L_0 = 0$, $L_1 = 0$, $L_2 = 0$, to determine the ratios of C_0, C_1, D_0, D_1.

28. Prove that the solution of the biquadratic does not involve the extraction of a cube root when any relation among the roots α, β, γ, δ exists which can be expressed by the vanishing of a rational function of a root θ of the reducing cubic.

Any rational function of θ can always be depressed to the second degree, as in the preceding example. Hence the determination of θ will not involve the extraction of a cube root; and the formula of Ex. 26 shows that the expression for the root of the biquadratic will not then involve any cube root.

29. Find in each case the relation which connects the roots of the biquadratic when the equation

$$4\rho^3 - I\rho + J = 0$$

is satisfied by any of the following values of ρ :—

$$(1)\ \frac{H}{a},\quad (2)\ c,\quad (3)\ 0,\quad (4)\ \frac{\sqrt{ae}-c}{2},\quad (5)\ \sqrt[3]{\frac{-J}{4}},\quad (6)\ \sqrt{\frac{I}{12}},\quad (7)\ \frac{3J}{2I},\quad (8)\ \frac{ad-bc}{2b}.$$

Ans. (1) $\beta + \gamma - \alpha - \delta = 0$ (2) $\beta + \gamma = 0$, (3) $(\gamma - \alpha)(\beta - \delta) - (\alpha - \beta)(\gamma - \delta) = 0$,

(4), (8) $\beta\gamma - \alpha\delta = 0$, (5) $(\gamma - \alpha)(\beta - \delta) - \omega(\alpha - \beta)(\gamma - \delta) = 0$, (6), (7) $\beta - \gamma = 0$.

30. Prove the identity

$$a_0^6(I^3 - 27J^2) \equiv (a_0^2 I - 3H^2)(a_0^2 I - 12H^2)^2 + 27G^2(G^2 + 2a_0^3 J).$$

This may be proved as follows:—Putting $a_1 = 0$ in the values of I and J, and expanding, it readily appears that the part of Δ independent of a_1 may be thrown into the form

$$a_0 a_4 (a_0 a_4 - 9a_2^2)^2 + 27 a_0 a_3^2 (2a_0 a_2 a_4 - a_0 a_3^2 - 2a_2^3).$$

Now, replacing a_2, a_3, a_4 by A_2, A_3, A_4, and substituting for the latter quantities the values of Art. 37, we obtain the result.—Mr. M. Roberts.

31. When a biquadratic has two equal roots, prove that Euler's cubic has two equal roots whose common value is

$$\frac{3aJ - 2HI}{2I};$$

and hence show that the remaining two roots of the biquadratic in this case are real, equal, or imaginary, according as $2HI - 3aJ$ is negative, zero, or positive.

32. Prove that when a biquadratic has—(1) two distinct pairs of equal roots the last two terms of the equation of squared differences (Art. 67) vanish, giving the conditions $\Delta = 0$, $2HI - 3aJ = 0$; and when it has—(2) three roots equal, the last three terms of this equation vanish, giving the conditions $I = 0$, $J = 0$; and show the equivalence of the conditions in the former case with those already obtained in Ex. 3, Art. 61, and Ex. 12, p. 148. Prove also that the equation of squared differences reduces in the former case to $\phi^2(a^2\phi + 12H)^4$, and in the latter case to $\phi^3(a^2\phi + 16H)^3$.

CHAPTER VII.

PROPERTIES OF THE DERIVED FUNCTIONS.

69. **Graphic Representation of the Derived Function.**—Let APB be the curve representing the polynomial $f(x)$, and P the point on it corresponding to any value of the variable $x = OM$. We proceed to determine the mode of representing the value of $f'(x)$ at the point P. Take a second point Q on the curve, corresponding to a value of x which exceeds OM by a small quantity h. Thus

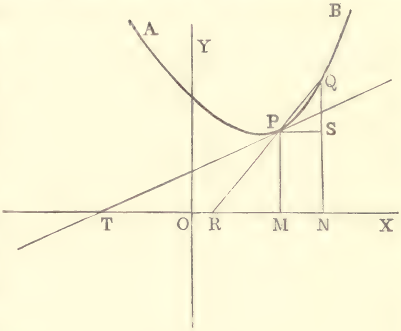

Fig. 5.

$$OM = x, \quad MN = h, \quad ON = x + h;$$

also $$PM = f(x), \quad QN = f(x + h).$$

The expansion of Art. 6 gives

$$f(x+h) = f(x) + f'(x)\,h + \frac{f''(x)}{1 \cdot 2}\,h^2 + \ldots\ldots,$$

or $$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x)}{1 \cdot 2}\,h + \ldots\ldots \qquad (1)$$

But $$\frac{f(x+h) - f(x)}{h} = \frac{QS}{MN} = \frac{QS}{PS} = \tan QPS = \tan PRN.$$

Now, when h is indefinitely diminished, the point Q approaches, and ultimately coincides with, P; the chord PQ becomes the

tangent PT to the curve at P; the angle PRN becomes PTM. Also all terms of the right-hand member of equation (1) except the first diminish indefinitely, and ultimately vanish when $h = 0$. The equation (1) becomes therefore

$$\tan PTM = f'(x);$$

from which we conclude that *the value assumed by the derived function $f'(x)$ on the substitution of any value of x is represented by the tangent of the angle made with the axis OX by the tangent at the corresponding point to the curve representing the function $f(x)$.*

70. **Maximum and Minimum Values of a Polynomial. Theorem.**—*Any value of x which renders $f(x)$ a maximum or minimum is a root of the derived equation $f'(x) = 0$.*

Let a be a value of x which renders $f(x)$ a *minimum*. We proceed to prove that $f'(x) = 0$. Let h represent a small increment or decrement of x. We have, since $f(a)$ is a minimum,

$$f(a) < f(a + h), \text{ also } f(a) < f(a - h);$$

hence $f(a + h) - f(a)$, and $f(a - h) - f(a)$ are both positive, *i.e.* the following two expressions are positive:—

$$f'(a)\,h + \frac{f''(a)}{1 . 2}h^2 + \ldots\ldots,$$

$$-f'(a)\,h + \frac{f''(a)}{1 . 2}h^2 - \ldots\ldots.$$

Now, when h is very small, we know (Art. 5) that the signs of these expressions are the same as the signs of their first terms; hence, in order that both should be positive, $f'(a)$ must vanish; and, moreover, *$f''(a)$ must be positive*. An exactly similar proof shows that when $f(a)$ is a *maximum* $f'(a) = 0$, *and $f''(a)$ is negative*. Thus, in order to find the maximum and minimum values of a polynomial $f(x)$, we must solve the equation $f'(x) = 0$, and substitute the roots in $f(x)$. Each root will furnish a maximum or minimum, the criterion to decide between these being the sign of $f''(x)$ when the root is substituted in it—*when $f''(x)$ is negative, the value is a maximum; and when $f''(x)$ is positive, the value is a minimum.*

The theorem of this Article follows at once from the construction of Art. 69; for it is plain that when the value of $f(x)$ is a maximum, as at P, P' (fig. 6), or a minimum, as at p, p', the tangent to the curve will be parallel to the axis OX, and, consequently,

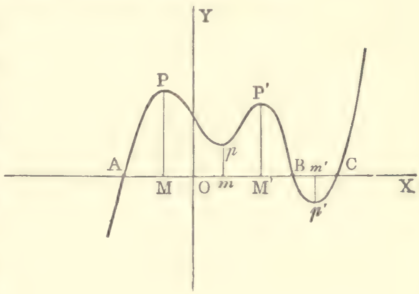

Fig. 6.

$$\tan PTM = f'(x) = 0.$$

Fig. 6 represents a polynomial of the 5th degree. Corresponding to the four roots of $f'(x) = 0$ (supposed all real in this case), viz. OM, Om, OM', Om', there are two maxima, MP, $M'P'$; and two minima, mp, $m'p'$.

EXAMPLES.

1. Find the max. or min. value of

$$f(x) \equiv 2x^2 + x - 6.$$

$$f'(x) = 4x + 1, \quad f''(x) = 4.$$

$$x = -\frac{1}{4} \text{ makes } f(x) = \frac{-49}{8}, \text{ a minimum.}$$

(See fig. 2, p. 15.)

2. Find the max. and min. values of

$$f(x) \equiv 2x^3 - 3x^2 - 36x + 14.$$

$$f'(x) = 6(x^2 - x - 6), \quad f''(x) = 6(2x - 1).$$

$$x = -2 \text{ makes } f(x) = 68, \text{ a maximum.}$$

$$x = 3 \text{ makes } f(x) = -67, \text{ a minimum.}$$

3. Find the max. and min. values of

$$f(x) \equiv 3x^4 - 16x^3 + 6x^2 - 48x + 7.$$

Here $f'(x) = 0$ has only one real root, $x = 4$; and it gives a minimum value, $f(x) = -345$.

4. Find the max. and min. values of

$$f(x) \equiv 10x^3 - 17x^2 + x + 6.$$

The roots of $f'(x)$ are, approximately, ·0302, 1·1031. The former gives a maximum value, the latter a minimum. (See fig. 3, p. 16.)

71. **Rolle's Theorem.**—*Between two consecutive real roots a and b of the equation $f(x) = 0$ there lies at least one real root of the equation $f'(x) = 0$.*

For as x increases from a to b, $f(x)$, varying continuously from $f(a)$ to $f(b)$, must begin by increasing and then diminish, or must begin by diminishing and then increase. It must, therefore, pass through at least one maximum or minimum value during the passage from $f(a)$ to $f(b)$. This value ($f(\alpha)$, suppose) corresponds to some value α of x between a and b, which by the theorem of Art. 70 is a root of the equation $f'(x) = 0$.

The figure in the preceding Article illustrates this theorem. We observe that between the two points of section A and B there are *three* maximum or minimum values, and between the two points B and C there is one such value. It appears also from the figure that the number of such values between two consecutive points of section of the axis is always odd.

Corollary.—*Two consecutive roots of the derived equation may not comprise between them any root of the original equation, and never can comprise more than one.*

The first part of this proposition merely asserts that between two adjacent zero values of a polynomial there may be several maxima and minima; and the second part follows at once from the above theorem; for if two consecutive roots of $f'(x) = 0$ comprised between them more than one root of $f(x) = 0$, we should then have two consecutive roots of this latter equation comprising between them no root of $f'(x) = 0$, which is contradictory to the theorem.

72. **Constitution of the Derived Functions.**—Let the roots of the equation $f(x) = 0$ be $a_1, a_2, a_3, \ldots a_n$. We have

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n).$$

In this identical equation substitute $y + x$ for x;

$$\begin{aligned} f(y + x) &= (y + x - a_1)(y + x - a_2) \ldots (y + x - a_n) \\ &= y^n + q_1 y^{n-1} + q_2 y^{n-2} + \ldots + q_{n-1} y + q_n, \end{aligned}$$

where

$$q_1 \;\;= x - a_1 + x - a_2 + x - a_3 + \ldots + x - a_n,$$

$$q_2 \;\;= (x - a_1)(x - a_2) + (x - a_1)(x - a_3) + \ldots + (x - a_{n-1})(x - a_n),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$q_{n-1} = (x - a_2)(x - a_3) \ldots (x - a_n) + (x - a_1)(x - a_3) \ldots (x - a_n) + \ldots$$
$$+ (x - a_1)(x - a_2) \ldots (x - a_{n-1}),$$

$$q_n \;\;= (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n).$$

We have, again,

$$f(y + x) = f(x) + f'(x)\,y + \frac{f''(x)}{1\,.\,2}\,y^2 + \ldots + y^n.$$

Equating the two expressions for $f(y + x)$, we obtain

$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$,

$f'(x) = (x - a_2)(x - a_3) \ldots (x - a_n) + \ldots$, as above written,

$\dfrac{f''(x)}{1\,.\,2}$ = the similar value of q_{n-2} in terms of x and the roots,

$$\cdots\cdots\cdots\cdots\cdots$$

The value of $f'(x)$ may be conveniently written as follows :—

$$f'(x) = \frac{f(x)}{x - a_1} + \frac{f(x)}{x - a_2} + \ldots + \frac{f(x)}{x - a_n}.$$

73. **Multiple Roots. Theorem.**—*A multiple root of the order m of the equation $f(x) = 0$ is a multiple root of the order $m - 1$ of the first derived equation $f'(x) = 0$.*

This follows immediately from the expression given for $f'(x)$ in the preceding Article; for if the factor $(x - a_1)^m$ occurs in $f(x)$, *i.e.* if $a_1 = a_2 = \ldots = a_m$; we have

$$f'(x) = \frac{mf(x)}{x - a_1} + \frac{f(x)}{x - a_{m+1}} + \ldots + \frac{f(x)}{x - a_n}.$$

Each term in this will still have $(x - a_1)^m$ as a factor, except the first, which will have $(x - a_1)^{m-1}$ as a factor; hence $(x - a_1)^{m-1}$ is a factor in $f'(x)$.

Cor. 1.—*Any root which occurs m times in the equation $f(x) = 0$ occurs in degrees of multiplicity diminishing by unity in the first $m-1$ derived equations.*

Since $f''(x)$ is derived from $f'(x)$ in the same manner as $f'(x)$ is from $f(x)$, it is evident by the theorem just proved that $f''(x)$ will contain $(x - a_1)^{m-2}$ as a factor. The next derived function, $f'''(x)$, will contain $(x - a_1)^{m-3}$; and so on.

Cor. 2.—*If $f(x)$ and its first $m-1$ derived functions all vanish for a value a of x, then $(x-a)^m$ is a factor in $f(x)$.*

This, which is the converse of the preceding corollary, is most readily established directly as follows:—Representing the derived functions by $f_1(x), f_2(x), \ldots . f_{m-1}(x)$ (see Art. 6), and substituting $a + x - a$ for x, we find that $f(x)$ may be expanded in the form

$$f(a) + f_1(a)(x-a) + \frac{f_2(a)}{1.2}(x-a)^2 + \ldots + \frac{f_{m-1}(a)}{1.2\ldots m-1}(x-a)^{m-1}$$

$$+ \frac{f_m(a)}{1.2\ldots m}(x-a)^m + \ldots + \frac{f_n(a)}{1.2\ldots n}(x-a)^n,$$

from which the proposition is manifest.

74. **Determination of Multiple Roots.**—It is easily inferred from the preceding Article that if $f(x)$ and $f'(x)$ have a common factor $(x-a)^{m-1}$, $(x-a)^m$ will be a factor in $f(x)$; for, by Cor. 1, the $m-2$ next succeeding derived functions vanish as well as $f(x)$ and $f'(x)$ when $x = a$; hence, by Cor. 2, a is a root of $f(x)$ of multiplicity m. In the same way it appears that if $f(x)$ and $f'(x)$ have other common factors

$$(x-\beta)^{p-1}, \ (x-\gamma)^{q-1}, \ (x-\delta)^{r-1}, \text{ \&c.},$$

the equation $f(x) = 0$ will have p roots equal to β, q roots equal to γ, r roots equal to δ, &c.

In order, therefore, to find whether any proposed equation has equal roots, and to determine such roots when they exist, we must find the greatest common measure of $f(x)$ and $f'(x)$. Let this be $\phi(x)$. The determination of the equal roots will depend on the solution of the equation $\phi(x) = 0$.

Examples.

1. Find the multiple roots of the equation

$$x^3 + x^2 - 16x + 20 = 0.$$

The G. C. M. of $f(x)$ and $f'(x)$ is easily found to be $x - 2$; hence $(x-2)^2$ is a factor in $f(x)$. The other factor is $x + 5$.

Whenever, after determining the multiple factors of $f(x)$, we wish to obtain the remaining factors, it will be found convenient to apply by repeated operations the method of division of Art. 8. Here, for example, we divide twice by $x - 2$, the calculation being represented as follows:—

$$\begin{array}{cccc}
1 & 1 & -16 & 20 \\
 & 2 & 6 & -20 \\
\hline
1 & 3 & -10 & 0 \\
 & 2 & 10 & \\
\cline{2-3}
1 & 5 & 0 &
\end{array}$$

Thus 1 and 5 being the two coefficients left, the third factor is $x + 5$. This operation verifies the previous result, the remainders after each division vanishing as they ought.

2. Find the multiple roots, and the remaining factor, of the equation

$$x^5 - 10x^2 + 15x - 6 = 0.$$

The G. C. M. of $f(x)$ and $f'(x)$ is found to be $x^2 - 2x + 1$. Hence $(x - 1)^3$ is a factor in $f(x)$. Dividing three times in succession by $x - 1$, we obtain

$$f(x) \equiv (x-1)^3 (x^2 + 3x + 6).$$

3. Find the multiple roots of the equation

$$x^4 - 2x^3 - 11x^2 + 12x + 36 = 0.$$

The G. C. M. of $f(x)$ and $f'(x)$ is $x^2 - x - 6$. The factors of this are $x + 2$ and $x - 3$. Hence

$$f(x) \equiv (x+2)^2 (x-3)^2.$$

4. Find all the factors of the polynomial

$$f(x) \equiv x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8.$$

Ans. $f(x) \equiv (x-1)(x+1)^2(x-2)^3$.

The ordinary process of finding the greatest common measure of a polynomial and its first derived function may become very laborious as the degree of the function increases. It is wrong, therefore, to speak, as is customary in works on the

Theory of Equations, of the determination in this way of the multiple roots of numerical equations as a simple process, and one preliminary to further investigations relative to the roots. It is chiefly in connexion with Sturm's theorem that the operation is of any practical value. The further consideration of multiple roots is deferred to Chap. IX., where this theorem will be discussed. It will be shown also in Chap. X., that the multiple roots of equations of degrees inferior to the sixth can, in any particular instance, be determined from simple considerations not involving the process of finding the greatest common measure.

75. This and the succeeding Article will be occupied with theorems which will be found of great importance in the subsequent discussion of methods of separating the roots of equations.

Theorem.—*In passing continuously from a value $a - h$ of x a little less than a real root a of the equation $f(x) = 0$ to a value $a + h$ a little greater, the polynomials $f(x)$ and $f'(x)$ have unlike signs immediately before the passage through the root, and like signs immediately after.*

Substituting $a - h$ in $f(x)$ and $f'(x)$, and expanding, we have

$$f(a-h) = f(a) - f'(a)\,h + \frac{f''(a)}{1 \cdot 2}\,h^2 - \ldots.$$

$$f'(a-h) = \qquad f'(a) - f''(a)\,h + \ldots\ldots$$

Now, since $f(a) = 0$, the signs of these expressions, depending on those of their first terms, are unlike. When the sign of h is changed, the signs of the expressions become the same. The theorem is therefore proved.

Corollary.—*The theorem remains true when a is a multiple root of any order of the equation $f(x) = 0$.*

Let the root be repeated r times. The following functions (using suffixes in place of the accents) all vanish :—

$$f(a),\ f_1(a),\ f_2(a),\ \ldots.\ f_{r-1}(a).$$

In the series for $f(a-h)$ and $f'(a-h)$ the first terms which do not vanish are, respectively,

$$\frac{f_r(a)}{1.2\ldots r}(-h)^r, \quad \frac{f_r(a)}{1.2\ldots r-1}(-h)^{r-1}.$$

These have plainly unlike signs; but when the sign of h is changed they will have like signs. Hence the proposition is established.

76. Extending the reasoning of the last Article to every consecutive pair of the series

$$f(x),\ f_1(x),\ f_2(x),\ \ldots f_{r-1}(x),$$

we may state the proposition generally as follows:—

Theorem.—*When any equation $f(x) = 0$ has an r-multiple root a, a value a little inferior to a gives to this series of r functions signs alternately positive and negative, or negative and positive; and a value a little superior to it gives to all these functions the same sign; and this sign is, moreover, the same as the sign of $f_r(a)$, the first derived function which does not vanish when a is substituted for x.*

In order to give a precise idea of the use of this theorem, let us suppose that $f_5(a)$ is the first function which does not vanish when a is substituted, and let its sign be negative; the conclusion which may be drawn from the theorem is, that for a value $a-h$ of x the signs of the series of functions $f, f_1, f_2, f_3, f_4, f_5$, are

$$+ - + - + -;$$

and for a value $a+h$ of x they are

$$- - - - - -;$$

for before the passage through the root the sign of f_4 must be different from that of f_5; the sign of f_3 must be different from that of f_4, and so on; and after the passage the signs of all the functions must be the same. It is of course assumed here that h is so small that no root of $f_5(x) = 0$ is included within the interval through which x travels.

Examples.

1. Find the multiple roots of the equation

$$f(x) \equiv x^4 + 12x^3 + 32x^2 - 24x + 4 = 0.$$

Ans. $f(x) \equiv (x^2 + 6x - 2)^2$.

2. Show that the binomial equation

$$x^n - a^n = 0$$

cannot have equal roots.

3. Show that the equation

$$x^n - nqx + (n-1)r = 0$$

will have a pair of equal roots if $q^n = r^{n-1}$.

4. Prove that the equation

$$x^5 + 5px^3 + 5p^2x + q = 0$$

has a pair of equal roots when $q^2 + 4p^5 = 0$; and that if it have one pair of equal roots it must have a second pair.

5. Apply the method of Art. 74 to determine the condition that the cubic

$$z^3 + 3Hz + G = 0$$

should have a pair of equal roots.

The last remainder in the process of finding the greatest common measure must vanish. *Ans.* $G^2 + 4H^3 = 0$.

6. Apply the same method to show that both G and H vanish when the cubic has three equal roots.

7. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic $f(x) = 0$, prove that

$$f'(\alpha) + f'(\beta) + f'(\gamma) + f'(\delta)$$

can be expressed as a product of three factors.

Ans. $(\alpha + \beta - \gamma - \delta)(\alpha + \gamma - \beta - \delta)(\alpha + \delta - \beta - \gamma)$.

8. If $\alpha, \beta, \gamma, \delta$, &c., be the roots of $f(x) = 0$, and α', β', γ', &c., of $f'(x) = 0$, prove

$$f'(\alpha) f'(\beta) f'(\gamma) f'(\delta) \ldots = n^n f(\alpha') f(\beta') f(\gamma') \ldots,$$

and that each is equal to the absolute term in the equation whose roots are the squares of the differences.

9. If the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$$

have a double root α, prove that α is a root of the equation

$$p_1 x^{n-1} + 2p_2 x^{n-2} + 3p_3 x^{n-3} + \ldots + np_n = 0.$$

10. Show that the max. and min. values of the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

are the roots of the equation

$$a^2\rho^2 - 2G\rho + \Delta = 0,$$

where Δ is the discriminant.

If the curve representing the polynomial $f(x)$ be moved parallel to the axis of y (see Art. 10) through a distance equal to a max. or min. value ρ, the axis of x will become a tangent to it, *i.e.* the equation $f(x) - \rho = 0$ will have equal roots. Hence the max. and min. values are obtained by forming the discriminant of $f(x) - \rho$, or by putting $d - \rho$ for d in $G^2 + 4H^3 = 0$.

11. Prove similarly that the max. and min. values of

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

are the roots of the equation

$$a^3\rho^3 - 3(a^2I - 9H^2)\rho^2 + 3(aI^2 - 18HJ)\rho - \Delta = 0,$$

where Δ is the discriminant of the quartic.

12. Apply the theorem of Art. 76 to the function

$$f(x) \equiv x^4 - 7x^3 + 15x^2 - 13x + 4.$$

We have

$$f_1(x) = 4x^3 - 21x^2 + 30x - 13,$$

$$f_2(x) = 2(6x^2 - 21x + 15),$$

$$f_3(x) = 2(12x - 21),$$

$$f_4(x) = 24.$$

Here $f_3(x)$ is the first function which does not vanish when $x = 1$; and $f_3(1)$ is negative. What the theorem proves is, that for a value a little less than 1 the signs of f, f_1, f_2, f_3 are $+ - + -$, and for a value a little greater than 1 they are all negative. We are able from this series of signs to trace the functions f, f_1, &c., in the neighbourhood of the point $x = 1$. Thus the curve representing $f(x)$ is above the axis before reaching the multiple point $x = 1$, and is below the axis immediately after reaching the point, and the axis must be regarded as cutting the curve in three coincident points, since $(x - 1)^3$ is a factor in $f(x)$ Again, the curve corresponding to $f_1(x)$ is below the axis both before and after the passage through the point $x = 1$. It touches the axis at that point. The curve representing $f_2(x)$ is above the axis before, and below the axis after, the passage, and cuts the axis at the point.

CHAPTER VIII.

LIMITS OF THE ROOTS OF EQUATIONS.

77. **Definition of Limits.**—In attempting to discover the real roots of numerical equations, it is in the first place advantageous to narrow the region within which they must be sought. We here take up the inquiry referred to in the observation at the end of Art. 4, and proceed to prove certain propositions relative to the limits of the real roots of equations.

A *superior limit* of the positive roots is any greater positive number than the greatest of them; an *inferior limit* of the positive roots is any smaller positive number than the smallest of them. A superior limit of the negative roots is any greater negative number than the greatest of them; an inferior limit of the negative roots is any smaller negative number than the smallest of them: the greatest negative number meaning here that nearest to $-\infty$.

When we have found limits within which all the real roots of an equation lie, the next step towards the solution of the equation is to discover the intervals in which the separate roots are situated. The principal methods in use for this latter purpose will form the subject of the next chapter.

The following Propositions all relate to the superior limits of the positive roots; to which, as will be subsequently proved, the determination of inferior limits and limits of the negative roots can be immediately reduced.

78. **Proposition I.**—*In any equation*

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0,$$

if the first negative term be $-p_rx^{n-r}$, *and if the greatest negative*

coefficient be $-p_k$, *then* $\sqrt[r]{p_k}+1$ *is a superior limit of the positive roots.*

Any value of x which makes

$$x^n > p_k\,(x^{n-r} + x^{n-r-1} + \ldots + x + 1) > p_k\,\frac{x^{n-r+1}-1}{x-1}$$

will, *à fortiori*, make $f(x)$ positive.

Now, taking x greater than unity, this inequality is satisfied by the following:—

$$x^n > p_k\,\frac{x^{n-r+1}}{x-1},$$

or

$$x^{n+1} - x^n > p_k x^{n-r+1},$$

or

$$x^{r-1}\,(x-1) > p_k,$$

which inequality again is satisfied by the following:—

$$(x-1)^{r-1}\,(x-1) = \text{or} > p_k,$$

clearly $x^r > p_k$

since plainly $x^{r-1} > (x-1)^{r-1}$. ~~no~~ yes

(Let $r=3$, $x=2$, $p_k=2$ AAA) $\therefore\ x > \sqrt[r]{p_k}$

We have, therefore, finally

$$(x-1)^r = \text{or} > p_k,$$

but x may be less than one so $x > \sqrt[r]{p_k}+1$ is an upper bound.

or

$$x = \text{or} > 1 + \sqrt[r]{p_k}.$$

your comment justs shows $\max(1, \sqrt[r]{p_k})$ works.

79. **Proposition II.**—*If in any equation each negative coefficient be taken positively, and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus formed increased by unity is a superior limit of the positive roots.*

Let the equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \ldots\ldots - a_rx^{n-r} + \ldots\ldots + a_n = 0,$$

in which, in order to fix our ideas, we regard the fourth coefficient as negative, and we consider also a negative coefficient in general, viz. $-a_r$.

Let each positive term in this equation be transformed by means of the formula

$$a_mx^m = a_m\,(x-1)\,(x^{m-1} + x^{m-2} + \ldots + x + 1) + a_m,$$

which is derived at once from

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \ldots + x + 1;$$

the negative terms remaining unchanged.

The polynomial $f(x)$ becomes then, the horizontal lines of the following corresponding to the successive terms of $f(x)$:—

$$\begin{array}{l}
a_0(x-1)x^{n-1} + a_0(x-1)x^{n-2} + a_0(x-1)x^{n-3} + \ldots + a_0(x-1)x^{n-r} + \ldots + a_0, \\
\qquad\qquad\qquad + a_1(x-1)x^{n-2} + a_1(x-1)x^{n-3} + \ldots + a_1(x-1)x^{n-r} + \ldots + a_1, \\
\qquad\qquad\qquad\qquad\qquad\qquad + a_2(x-1)x^{n-3} + \ldots + a_2(x-1)x^{n-r} + \ldots + a_2, \\
\qquad\qquad\qquad\qquad\qquad\qquad - a_3 x^{n-3}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - a_r x^{n-r}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_n.
\end{array}$$

We now regard the vertical columns of this expression as successive terms in the polynomial; the successive coefficients of x^{n-1}, x^{n-2}, &c., being

$$a_0(x-1), \quad (a_0 + a_1)(x-1), \quad (a_0 + a_1 + a_2)(x-1) - a_3, \quad \&c.$$

Any value of x greater than unity is sufficient to make positive every term in which no negative coefficient a_3, a_r, &c., occurs. To make the latter terms positive, we must have

$$(a_0 + a_1 + a_2)(x - 1) > a_3,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$(a_0 + a_1 + a_2 + \ldots + a_{r-1})(x-1) > a_r, \quad \&c.$$

Hence

$$x > \frac{a_3}{a_0 + a_1 + a_2} + 1, \; \ldots \; x > \frac{a_r}{a_0 + a_1 + a_2 + a_4 + \ldots + a_{r-1}} + 1, \; \&c.$$

And to ensure every term being made positive, we must take the value of the greatest of the quantities found in this way. Such a value of x, therefore, is a superior limit of the positive roots.

80. **Practical Applications.**—The propositions in the two preceding Articles furnish the most convenient *general* methods of finding in practice tolerably close limits of the roots. Sometimes one of the propositions will give the closer limit; sometimes the other. It is well, therefore, to apply both methods, and take the smaller limit. Prop. I. will usually be found the more advantageous when the first negative coefficient is preceded by several positive coefficients, so that r is large; and Prop. II. when large positive coefficients occur before the first large negative coefficient. In general, Prop. II. will give the closer limit. We speak of the integer next above the numerical value given by either proposition as the limit.

Examples.

1. Find a superior limit of the positive roots of the equation

$$x^4 - 5x^3 + 40x^2 - 8x + 23 = 0.$$

Prop. I. gives $8 + 1$, or 9, as limit.

Prop. II. gives $\frac{5}{1} + 1$, or 6. Hence 6 is a superior limit.

2. Find a superior limit of the positive roots of the equation

$$x^5 + 3x^4 + x^3 - 8x^2 - 51x + 18 = 0.$$

Prop. I. gives $\sqrt[3]{51} + 1$; and 5 is, therefore, a limit.

Prop. II. gives $\frac{51}{1 + 3 + 1} + 1$, and 12 is a limit.

In this case Prop. I. gives the closer limit.

3. Find a superior limit of the positive roots of

$$x^7 + 4x^6 - 3x^5 + 5x^4 - 9x^3 - 11x^2 + 6x - 8 = 0.$$

Of the fractions

$$\frac{3}{1 + 4}, \quad \frac{9}{1 + 4 + 5}, \quad \frac{11}{1 + 4 + 5}, \quad \frac{8}{1 + 4 + 5 + 6},$$

the third is the greatest, and Prop. II. gives the limit 3. Prop. I. gives 5.

4. Find a superior limit of the positive roots of

$$x^8 + 20x^7 + 4x^6 - 11x^5 - 120x^4 + 13x - 25 = 0.$$

Ans. Both methods give the limit 6.

5. Find a superior limit of the positive roots of

$$4x^5 - 8x^4 + 22x^3 + 98x^2 - 73x + 5 = 0.$$

Ans. Prop. I. gives 20. Prop. II. gives 3.

It is usually possible to determine by inspection a limit closer than that given by either of the preceding propositions. This method consists in arranging the terms of an equation in groups having a positive term first, and then observing what is the lowest integral value of x which will have the effect of rendering each group positive. The form of the equation will suggest the arrangement in any particular case.

6. The equation of Ex. 2 can be arranged as follows :—

$$x^2(x^3 - 8) + x(3x^3 - 51) + x^3 + 18 = 0.$$

$x = 3$, or any greater number, renders each group positive ; hence 3 is a superior limit.

7. The equation of Ex. 4 may be arranged thus :—

$$x^5(x^3 - 11) + 20x^4(x^3 - 6) + 4x^6 + 13x - 25 = 0.$$

$x = 3$, or any greater number, renders each group positive ; hence 3 is a limit.

8. Find a superior limit of the roots of the equation

$$x^4 - 4x^3 + 33x^2 - 2x + 18 = 0.$$

This can be arranged in the form

$$x^2(x^2 - 4x + 5) + 28x(x - \tfrac{1}{14}) + 18 = 0.$$

Now the trinomial $x^2 - 4x + 5$, having imaginary roots, is positive for all values of x (Art. 12). Hence $x = 1$ is a superior limit.

The introduction in this way of a quadratic whose roots are imaginary, or of one with equal roots, will often be found useful.

9. Find a superior limit of the roots of the equation

$$5x^5 - 7x^4 - 10x^3 - 23x^2 - 90x - 317 = 0.$$

In examples of this kind it is convenient to distribute the highest power of x among the negative terms. Here the equation may be written

$$x^4(x - 7) + x^3(x^2 - 10) + x^2(x^3 - 23) + x(x^4 - 90) + x^5 - 317 = 0,$$

so that 7 is evidently a superior limit of the roots. In this case the general methods give a very high limit.

10. Find a superior limit of the roots of the equation

$$x^4 - x^3 - 2x^2 - 4x - 24 = 0.$$

When there are several negative terms, and the coefficient of the highest term unity, it is convenient to multiply the whole equation by such a number as will enable us to distribute the highest term among the negative terms. Here, multiplying by 4, we can write the equation as follows :—

$$x^3(x - 4) + x^2(x^2 - 8) + x(x^3 - 16) + x^4 - 96 = 0,$$

and 4 is a superior limit. The general methods give 25.

81. **Proposition III.**—*Any number which renders positive the polynomial $f(x)$ and all its derived functions $f_1(x), f_2(x), \ldots f_n(x)$ is a superior limit of the positive roots of the equation $f(x) = 0$.*

This method of finding limits is due to Newton. It is much more laborious in its application than either of the preceding methods; but it has the advantage of giving always very close limits; and in the case of an equation all whose roots are real the limit found in this way is, as will be subsequently proved, the next integer above the greatest positive root.

To prove the proposition, let the roots of the equation $f(x) = 0$ be diminished by h; then $x - h = y$, and

$$f(y + h) = f(h) + f_1(h)\,y + \frac{f_2(h)}{1.2}\,y^2 + \ldots + \frac{f_n(h)}{1.2\ldots n}\,y^n.$$

If now h be such as to make all the coefficients

$$f(h),\ f_1(h),\ f_2(h),\ \ldots f_n(h)$$

positive, the equation in y cannot have a positive root; that is to say, the equation in x has no root greater than h; hence h is a superior limit of the positive roots.

EXAMPLE.

$$f(x) \equiv x^4 - 2x^3 - 3x^2 - 15x - 3.$$

In applying Newton's method of finding limits to any example the general mode of procedure is as follows:—Take the smallest integral number which renders $f_{n-1}(x)$ positive; and proceeding upwards in order to $f_1(x)$, try the effect of substituting this number for x in the other functions of the series. When any function is reached which becomes negative for the integer in question, increase the integer successively by units, till it makes that function positive; and then proceed with the new integer as before, increasing it again if another function in the series should become negative; and so on, till an integer is reached which renders all the functions in the series positive. In the present example the series of functions is

$$f(x) = x^4 - 2x^3 - 3x^2 - 15x - 3,$$

$$f_1(x) = 4x^3 - 6x^2 - 6x - 15,$$

$$\tfrac{1}{2}f_2(x) = 6x^2 - 6x - 3.$$

$$\tfrac{1}{6}f_3(x) = 4x - 2,$$

$$\tfrac{1}{24}f_4(x) = 1.$$

Here $x = 1$ makes $f_3(x)$ positive. We try then the effect of the substitution $x = 1$ in $f_2(x)$. It makes $f_2(x)$ negative. Increase by 1; and $x = 2$ makes $f_2(x)$ positive. Try the effect of $x = 2$ in $f_1(x)$; it gives a negative result. Increase by 1: and $x = 3$ makes $f_1(x)$ positive. Proceeding upwards, the substitution $x = 3$ makes $f(x)$ negative; and increasing again by unity, we find that $x = 4$ makes $f(x)$ positive. Hence 4 is the superior limit required.

It is assumed in this mode of applying Newton's rule, that when any number makes all the derived functions up to a certain stage positive, any higher number will also make them positive; so that there is no occasion to try the effect of the higher number on the functions in the series below that one where our upward progress is arrested. This is evident from the equation

$$\phi(a + h) = \phi(a) + \phi'(a)h + \phi''(a)\frac{h^2}{1.2} + \ldots$$

(taking $\phi(x)$ to represent any function in the series, and using the common notation for derived functions), which shows that if $\phi(a)$, $\phi'(a)$, $\phi''(a)$, ... are all positive, and h also positive, $\phi(a + h)$ must be positive.

It may be observed that one advantage of Newton's method is that often, as in the present instance, it gives us a knowledge of the two successive integers between which the highest root lies. Thus in the present example, since $f(x)$ is negative for $x = 3$, and positive for $x = 4$, we know that the greatest root of the equation lies between 3 and 4.

82. **Inferior Limits, and Limits of the Negative Roots.**—To find an inferior limit of the positive roots, the equation must be first transformed by the substitution $x = \frac{1}{y}$. Find then a superior limit h of the positive roots of the equation in y. The reciprocal of this, viz. $\frac{1}{h}$, will be the required inferior limit; for since

$$y < h, \quad \frac{1}{y} > \frac{1}{h}, \quad \text{i.e. } x > \frac{1}{h}.$$

To find limits of the negative roots, we have only to transform the equation by the substitution $x = -y$. This transformation changes the negative into positive roots. Let the superior and inferior limits of the positive roots of the equation in y be h and h'. Then $-h$ and $-h'$ are the limits of the negative roots of the proposed equation.

83. **Limiting Equations.**—*If all the real roots of the equation $f'(x) = 0$ could be found, it would be possible to determine the number of real roots of the equation $f(x) = 0$.*

To prove this, let the real roots of $f'(x) = 0$ be, in ascending order of magnitude, $\alpha', \beta', \gamma', \ldots \lambda'$; and let the following series of values be substituted for x in $f(x)$:—

$$-\infty, \alpha', \beta', \gamma', \ldots \lambda', +\infty.$$

When any successive two of these quantities give results with different signs there is a root of $f(x) = 0$ between them; and by the Cor., Art. 71, there is only one; and when they give results with the same sign there is, by the same Cor., no root between them. Thus each change of sign in the results of the successive substitutions proves the existence of one real root of the proposed equation.

If all the roots of $f(x) = 0$ are real, it is evident, by the theorem of Art. 71, that all the roots of $f'(x) = 0$ are also real, and that they lie one by one between each adjacent pair of the roots of $f(x) = 0$. In the same case, and by the same theorem, it follows that the roots of $f''(x) = 0$, and of all the successive derived functions, are real also; and the roots of any function lie severally between each adjacent pair of the roots of the function from which it is immediately derived.

Equations of this kind, which are one degree below the degree of any proposed equation, and whose roots lie severally between each adjacent pair of the roots of the proposed, are called *limiting equations.*

It is evident that in the application of Newton's method of finding limits of the roots, when the roots of $f(x) = 0$ are all real, in proceeding according to the method explained in Art. 81, the function $f(x)$ is itself the last which will be rendered positive, and therefore the superior limit arrived at is the integer next above the greatest root.

Examples.

1. Prove that any derived equation $f_m(x) = 0$ cannot have more imaginary roots, but may have more real roots, than the equation $f(x) = 0$ from which it is derived.

From this it follows that if any of the derived functions be found to have imaginary roots, the same number at least of imaginary roots must enter the primitive equation.

2. Apply the method of Art. 83 to determine the conditions that the equation

$$x^3 - qx + r = 0$$

should have all its roots real.

3. Determine by the same method the nature of the roots of the equation

$$x^n - nqx + (n-1)r = 0.$$

Ans. When n is even, the equation has two real roots or none, according as $q^n >$ or $< r^{n-1}$.

When n is odd, the equation has three real roots or one, according as $q^n >$ or $< r^{n-1}$.

4. The equation $x^n(x-1)^n = 0$ has all its roots real; hence show, by forming the n^{th} derived function, that the following equation has all its roots real and unequal, and situated between 0 and 1:—

$$x^n - n\frac{n}{2n}x^{n-1} + \frac{n(n-1)}{1.2}\,\frac{n(n-1)}{2n(2n-1)}x^{n-2} - \&c. = 0.$$

5. Show similarly by forming the n^{th} derived of $(x^2-1)^n$ that the following equation has all its roots real and unequal, and situated between -1 and 1:—

$$x^n - n\frac{n(n-1)}{2n(2n-1)}x^{n-2} + \frac{n(n-1)}{1.2}\,\frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)}x^{n-4} - \&c. = 0.$$

6. If any two of the quantities, l, m, n in the following equation be put equal to zero, show that the quadratic to which the equation then reduces is a limiting equation; and hence prove that the roots of the proposed are all real:—

$$(x-a)(x-b)(x-c) - l^2(x-a) - m^2(x-b) - n^2(x-c) - 2lmn = 0.$$

7. Discuss the nature of the roots of the equation

$$x^4 + 4x^3 - 2x^2 - 12x + p = 0,$$

according to the different values of p.

Apply Art. 83. When p is less than -7, two roots are real and two imaginary; when p lies between -7 and 9, all the roots are real; and when p is greater than 9, the roots are all imaginary. The equation has two equal roots when $p = -7$, and two pairs of equal roots when $p = 9$.

CHAPTER IX.

SEPARATION OF THE ROOTS OF EQUATIONS.

84. By the methods of the preceding chapter we are enabled to find limits between which all the real roots of any numerical equation lie. Before proceeding to the actual approximation to any particular root, it is necessary to separate the interval in which it is situated from the intervals which contain the remaining roots. The present chapter will be occupied with certain theorems whose object is to determine the number of real roots between any two arbitrarily assumed values of the variable. It is plain that if this object can be effected, it will then be possible to tell not only the total number of real roots, but also the limits within which the roots separately lie.

The theorems given for this purpose by Fourier and Budan, although different in statement, are identical in principle. For purposes of exposition Fourier's statement is the more convenient, while with a view to practical application the statement of Budan will be found superior. The theorem of Sturm, although more laborious in practice, has the advantage over the preceding that it is unfailing in its application, giving always the exact number of real roots situated between any two proposed quantities; whereas the theorem of Fourier and Budan gives only a certain limit which the number of real roots in the proposed interval cannot exceed.

85. **Theorem of Fourier and Budan.**—*Let two numbers a and b, of which a is the less, be substituted in the series formed by $f(x)$ and its successive derived functions, viz.,*

$$f(x), f_1(x), f_2(x), \ldots, f_n(x);$$

the number of real roots which lie between a and b cannot be greater than the excess of the number of changes of sign in the series when a is substituted for x, over the number of changes when b is substituted for x; and when the number of real roots in the interval falls short of that difference, it will be by an even number.

This is the form in which Fourier states the theorem.

It is to be understood here, as elsewhere, that, when we speak of two numbers a and b, of which a is the less, one or both of them may be negative, and what is meant is that a is nearer than b to $-\infty$.

We proceed to examine the changes which may occur among the signs of the functions in the above series, the value of x being supposed to increase continuously from a to b. The following different cases can arise:—

(1). The value of x may pass through a single root of the equation $f(x) = 0$.

(2). It may pass through a root occurring r times in $f(x) = 0$.

(3). It may pass through a root of one of the auxiliary functions $f_m(x) = 0$, this root not occurring in either $f_{m-1}(x) = 0$ or $f_{m+1}(x) = 0$.

(4). It may pass through a root occurring r times in $f_m(x) = 0$, and not occurring in $f_{m-1}(x) = 0$.

In what follows the symbol x is omitted after f for convenience.

(1). In the first case it is evident, from Art. 75, that in passing through a root of the equation $f(x) = 0$, one change of sign is lost; for f and f_1 have unlike signs immediately before, and like signs immediately after, the passage through the root.

(2). In the second case, in passing through an r-multiple root of $f(x) = 0$, it is evident that r changes of sign are lost; for, by Art. 76, immediately before the passage the series of functions

$$f, f_1, f_2, \ldots f_{r-1}, f_r$$

have signs alternately $+$ and $-$, or $-$ and $+$, and immediately after the passage have all the same sign as f_r.

(3). In the third case, the root of $f_m(x) = 0$ must give to f_{m-1} and f_{m+1} either like signs or unlike signs. Suppose it to give like signs; then in passing through the root two changes of sign are lost, for before the passage the sign of f_m is different from these like signs, and after the passage it is the same (Art. 76). Suppose it to give unlike signs; then no change of sign is lost, for before the passage the signs of f_{m-1}, f_m, f_{m+1} must be either $+ \; + \; -$, or $- \; - \; +$, and after the passage these become $+ \; - \; -$, and $- \; + \; +$. On the whole, therefore, we conclude that no variation of sign can be gained, but two variations may be lost, on the passage through a root of $f_m(x) = 0$.

(4). In the fourth case x passes through a value (let us say a) which causes not only f_m but also $f_{m+1}, f_{m+2}, \ldots, f_{m+r-1}$ to vanish. It is evident from the theorem of Art. 76 that during the passage a number of changes of sign will always be lost. The definite number may be collected by considering the series of functions

$$f_{m-1}, f_m, f_{m+1}, \ldots, f_{m+r-1}, f_{m+r}.$$

We easily obtain the following results:—

(a). When $f_{m-1}(a)$ and $f_{m+r}(a)$ have like signs:

If r be even, r changes are lost.
If r be odd, $r + 1$ changes are lost.

(b). When $f_{m-1}(a)$ and $f_{m+r}(a)$ have unlike signs:

If r be even, r changes are lost.
If r be odd, $r - 1$ changes are lost.

We conclude, therefore, on the whole, that an even number of changes is lost during the passage through an r-multiple root of $f_m(x)$.

It will be observed that (1) is a particular case of (2), and (3) of (4), *i.e.* when $r = 1$. Since, however, the cases (1) and (3) are those of ordinary occurrence, it is well to give them a separate classification.

Reviewing the above proof, we conclude that as x increases from a to b no change of sign can be gained; that for each

passage through a single root of $f(x) = 0$ one change is lost; and that under no circumstances except a passage through a root of $f(x) = 0$ can an odd number of changes be lost. Hence the number of changes lost during the whole variation of x from a to b must be either equal to the number of real roots of $f(x) = 0$ in the interval, or must exceed it by an even number. The theorem is therefore proved.

86. **Application of the Theorem.**—The form in which the theorem has been stated by Budan is, as has been already observed, more convenient for practical purposes than that just given. It is as follows:—*Let the roots of an equation $f(x) = 0$ be diminished, first by a and then by b, where a and b are any two numbers of which a is the less; then the number of real roots between a and b cannot be greater than the excess of the number of changes of sign in the first transformed equation over the number in the second.*

This is evidently included in Fourier's statement, for the two transformed equations are (see Art. 33)—

$$f(a) + f_1(a)y + \frac{f_2(a)}{1.2}y^2 + \ldots + \frac{f_n(a)}{1.2\ldots n}y^n = 0,$$

$$f(b) + f_1(b)y + \frac{f_2(b)}{1.2}y^2 + \ldots + \frac{f_n(b)}{1.2\ldots n}y^n = 0;$$

from which, assuming the results of the last Article, the above proposition is manifest.

The reason why the theorem in this form is convenient in practice is, that we can apply the expeditious method of diminishing the roots given in Art. 33.

Examples.

1. Find the situations of the roots of the equation

$$x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$$

We shall examine this function for values of x between the intervals

$$-10, \quad -1, \quad 0, \quad 1, \quad 10;$$

these numbers being assumed on account of the facility of calculation. Diminution

of the roots by 1 gives the following series of coefficients of the transformed equation :—

$$1, \quad 2, \quad -26, \quad 15, \quad 65, \quad -78.$$

In diminishing the roots by 10, it is apparent at the very outset of the calculation that the signs of the coefficients of the transformed equation will be all positive ; so that there is no occasion to complete the calculation in this case.

In diminishing the roots by -10 and -1, it is convenient to change the alternate signs of the equation, and diminish the roots by $+10$ and $+1$; and then in the result change the alternate signs again. The coefficients of the transformed equation when the roots are diminished by -1 are

$$1, \quad -8, \quad -2, \quad 139, \quad -291, \quad 60.$$

In diminishing by -10 we observe in the course of the operation, as before, that the signs will be all positive in the result, *i.e.* when the alternate signs are changed they will be alternately positive and negative.

Hence we have the following scheme :—

$$\begin{array}{llllllll}
(-10) & + & - & + & - & + & - & \\
(-1) & + & - & - & + & - & + & \\
(0) & + & - & - & + & - & -, & \text{the equation itself.} \\
(1) & + & + & - & + & + & - & \\
(10) & + & + & + & + & + & + &
\end{array}$$

These signs are the signs taken by $f(x)$ and the several derived functions f_1, f_2, f_3, f_4, f_5 on the substitution of the proposed numbers ; but it is to be observed that they are here written, not in the order of Art. 85, but in the reverse order, viz. $f_5, f_4, f_3, f_2, f_1, f$.

From these we draw the following conclusions :—All the real roots must lie between -10 and $+10$; one real root lies between -10 and -1, since one change of sign is lost ; one real root lies between -1 and 0, since one change of sign is lost ; no real root lies between 0 and 1 ; and between 1 and 10, since three changes of sign are lost, there is at least one real root ; but we are left in doubt as to the nature of the other two roots : whether they are imaginary, or whether there are three real roots between 1 and 10.

We might proceed to examine, by further transformations, the interval between 1 and 10 more closely, in order to determine the nature of the two doubtful roots ; but it is evident that the calculations for this purpose might, if the roots were nearly equal, become very laborious. This is the weak side of the theorem of Fourier and Budan. Both writers have attempted to supply this defect, and have given methods of determining the nature of the roots in doubtful intervals ; but as these methods are complicated, we do not stop to explain them ; the more especially as the theorem of Sturm effects fully the purposes for which the supplementary methods of Fourier and Budan were invented.

2. Analyse the equation of Ex. 1, p. 100, viz.,

$$x^3 + x^2 - 2x - 1 = 0.$$

The roots of this are all real, and lie between -2 and 2 (see Ex. 5, p. 100). Whenever the roots of an equation are all real, the signs of Fourier's functions determine the exact number of real roots between any two proposed integers. We obtain the following result:—The roots lie in the intervals

$$(-2, -1); \quad (-1, 0); \quad (1, 2).$$

3. Analyse the equation of Ex. 3, p. 100, viz.,

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0.$$

Ans. Two roots in the interval $(-2, -1)$, and one root in each of the intervals $(-1, 0)$; $(0, 1)$; $(1, 2)$.

4. Analyse the equation

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0.$$

The equation can have no negative roots. Diminish the roots by 10 several times in succession till the signs of the coefficients become all positive. We obtain the following result:—

(0)	+	−	+	−	+
(10)	+	−	+	+	
(20)	+	0	−	+	+
(30)	+	+	+	−	+
(40)	+	+	+	+	+

Thus, there is one root between 0 and 10, and one between 10 and 20; no root between 20 and 30. Between 30 and 40 either there are two real roots, or there is an indication of a pair of imaginary roots. That the former is the case will appear by diminishing the roots of the third transformed equation by units. This process will separate the roots, which will be found to lie between (2, 3) and (4, 5); so that the proposed equation has a third real root in the interval (32, 33), and a fourth in the interval (34, 35).

87. **Application of the Theorem to Imaginary Roots.**—Since there exist only n changes of sign to be lost in the passage of x from $-\infty$ to $+\infty$, if we have any reason for knowing that a pair of changes is lost during the passage of x through an interval which includes no real root of the equation, we may be assured of the existence of a pair of imaginary roots. Circumstances of this nature will arise in the application of Fourier's theorem when any of the transformed equations contain vanishing coefficients. For we can assign by the principle of Art. 76 the proper sign to this coefficient, corresponding to

values of x immediately before and immediately after that value which causes the coefficient to vanish; the whole interval being so small that it may be supposed not to include any root of the equation $f(x) = 0$.

Examples.

1. Analyse the equation

$$f(x) \equiv x^4 - 4x^3 - 3x + 23 = 0.$$

We shall examine this function between the intervals 0, 1, 10. The transformed equations are

$$\tfrac{1}{24}f_4(0)\,x^4 + \tfrac{1}{6}f_3(0)\,x^3 + \tfrac{1}{2}f_2(0)\,x^2 + f_1(0)\,x + f(0) = 0,$$

$$\tfrac{1}{24}f_4(1)\,x^4 + \tfrac{1}{6}f_3(1)\,x^3 + \tfrac{1}{2}f_2(1)\,x^2 + f_1(1)\,x + f(1) = 0,$$

$$\tfrac{1}{24}f_4(10)\,x^4 + \tfrac{1}{6}f_3(10)\,x^3 + \tfrac{1}{2}f_2(10)\,x^2 + f_1(10)\,x + f(10) = 0,$$

the first of these being the proposed equation itself.

Making the calculations by the method of the preceding Article, we find that the coefficient $f_3(1) = 0$, and we have the following scheme :—

(0)	+	−	0	−	+
(1)	+	0	−	−	+
(10)	+	+	+	+	+

We may now replace each of the rows containing a zero coefficient by two, the first corresponding to a value a little less, and the second to a value a little greater, than that which gives the zero coefficients; the signs being determined by the principle established in Art. 76. It must be remembered that in the above scheme the signs representing the derived functions are written in the reverse order to that of the Article referred to. The scheme will then stand as follows, using h to represent a very small positive quantity :—

(0)	$-h$	+	−	+	−	+
	$+h$	+	−	−	−	+
(1)	$1-h$	+	−	−	−	+
	$1+h$	+	+	−	−	+
(10)		+	+	+	+	+

In this scheme the signs corresponding to $-h$ and $+h$ are determined by the condition that the sign of the coefficient which is zero when $x = 0$ must, when $x = -h$, be different from that next to it on the left-hand side; and when $x = +h$ it must be the same. The signs corresponding to $1 - h$ and $1 + h$ are determined in a similar manner.

Now since a pair of changes is lost in the interval $(-h, +h)$, and since the equation has no real root between $-h$ and $+h$, we have proved the existence of a pair of imaginary roots. Two changes of sign are lost between $1+h$ and 10, so that this interval either includes a pair of real roots, or presents an indication of a pair of imaginary roots. Which of these is the case remains still doubtful.

2. If several coefficients vanish, we may be able to establish the existence of several pairs of imaginary roots. This will appear from the following example:—

$$x^6 - 1 = 0.$$

The signs corresponding to $-h$ and $+h$ are, by the theorem of Art. 76,

$(-h)$	+	−	+	−	+	−	−
$(+h)$	+	+	+	+	+	+	−

Hence, since no root exists between $-h$ and $+h$, and since 4 changes of sign are lost in passing from a value very little less than 0 to one very little greater, we are assured of the existence of two pairs of imaginary roots. The other two roots are in this case plainly real (see Art. 14).

The number of imaginary roots in any binomial equation can be determined in this way.

3. Find the character of the roots of the equation

$$x^8 + 10x^3 + x - 4 = 0.$$

In passing from a small negative to a small positive value of x we obtain the following series of signs:—

$(-h)$	+	−	+	−	+	+	−	+	−
(0)	+	0	0	0	0	+	0	+	−
$(+h)$	+	+	+	+	+	+	+	+	−

Since six changes of sign are here lost, there are six imaginary roots. The remaining two roots are, by Art. 14, real: one positive, and the other negative. The negative root lies between -2 and -1, and the positive between 0 and 1.

4. Analyse completely the equation

$$x^6 - 3x^2 - x + 1 = 0.$$

There are two imaginary roots. Whenever, as in the present instance, the roots are comprised within small limits, it is convenient to diminish by successive units. In this way we find here a root between 0 and 1, and another between 1 and 2. Proceeding to negative roots, we find on diminishing by -1 that -1 is itself a root, and writing down the signs corresponding to a value a little greater than -1, we observe an indication of a second negative root between -1 and 0.

5. Analyse the equation

$$x^5 + x^4 + x^2 - 25x - 36 = 0.$$

There are two imaginary roots; one real positive root between 2 and 3; and two real negative roots in the intervals $(-3, -2)$, $(-2, -1)$.

88. **Corollaries from the Theorem of Fourier and Budan.**—The method of detecting the existence of imaginary roots explained in the preceding Article is called *The Rule of the Double Sign.* A similar rule, due to *De Gua*, was in use before the discovery of Fourier's theorem. This rule and Descartes' *Rule of Signs* are immediate corollaries from the theorem, as we proceed to show.

Cor. 1.—*De Gua's Rule for finding Imaginary Roots.*

The rule may be stated generally as follows :—*When* $2m$ *successive terms of an equation are absent, the equation has* $2m$ *imaginary roots; and when* $2m+1$ *successive terms are absent, the equation has* $2m+2$, *or* $2m$ *imaginary roots, according as the two terms between which the deficiency occurs have like or unlike signs.* This follows, as in case (4), Art. 85, by examining the number of changes of sign lost during the passage of x from a small negative value $-h$ to a small positive value h.

Cor. 2.—*Descartes' Rule of Signs.*

When 0 is substituted for x in the series of functions $f_n(x), f_{n-1}(x), \ldots f_2(x), f_1(x), f(x)$, the signs are the same as the signs of the coefficients $a_0, a_1, a_2, \ldots a_{n-1}, a_n$, of the proposed equation; and when $+\infty$ is substituted the signs are all positive. Fourier's theorem asserts that the number of roots between these limits, viz. the number of positive roots, cannot exceed the number of variations lost during the passage from 0 to $+\infty$, that is the number of changes of sign in the series $a_0, a_1, a_2 \ldots a_n$. This is Descartes' rule for positive roots; and the similar rule for negative roots follows in the usual way by changing the negative into positive roots.

Cor. 3.—*Newton's Method of finding Limits.*

When a number h has been found which renders positive each of the functions $f_n(x), f_{n-1}(x), \ldots f_2(x), f_1(x), f(x)$; since $+\infty$ also renders each of them positive, it follows from Fourier's theorem that there can be no root between h and $+\infty$, that is to say, h is a superior limit of the positive roots; and this is Newton's proposition (Art. 81).

89. **Sturm's Theorem.**—We have already shown (Art. 74) that it is possible by performing the common algebraical operation of finding the greatest common measure of a polynomial $f(x)$ and its first derived polynomial to find the equal roots of the equation $f(x) = 0$. Sturm has employed the same operation for the formation of the auxiliary functions which enter into his method of separating the roots of an equation.

Let the process of finding the greatest common measure of $f(x)$ and its first derived be performed. The successive remainders will go on diminishing in degree till we reach finally either one which divides that immediately preceding without remainder, or one which does not contain the variable at all, *i. e.* which is numerical. The former is, as we have already seen, the case of equal roots. The latter is the case where no equal roots exist. It is convenient to divide the discussion of Sturm's theorem into these two cases. We shall in the present Article consider the case where no equal roots exist; and proceed in the next Article to the case of equal roots. The performance of the operation itself will of course disclose the class to which any particular example is to be referred.

The auxiliary functions employed by Sturm are not the remainders as they present themselves in the calculation, but the remainders *with their signs changed.* In finding the greatest common measure of two expressions it is indifferent whether the signs of the remainders are changed or not: in the formation of Sturm's auxiliary functions the change is essential. It is convenient in practice to change the sign of each remainder before making it the next divisor.

Confining our attention for the present, therefore, to the case where no equal roots exist, Sturm's theorem may be stated as follows:—

Theorem.—*Let any two real quantities a and b be substituted for x in the series of $n + 1$ functions*

$$f(x),\ f_1(x),\ f_2(x),\ f_3(x),\ \ldots,\ f_{n-1}(x),\ f_n(x),$$

consisting of the given polynomial $f(x)$, its first derived $f_1(x)$, and

the successive remainders (with their signs changed) in the process of finding the greatest common measure of $f(x)$ and $f_1(x)$; then the difference between the number of changes of sign in the series when a is substituted for x, and the number when b is substituted for x expresses exactly the number of real roots of the equation $f(x) = 0$ between a and b.

The mode of formation of Sturm's functions supplies the following series of equations, in which $q_1, q_2, \ldots q_{n-1}$ represent the successive quotients in the operation :—

$$\left.\begin{aligned} f(x) &= q_1 f_1(x) - f_2(x), \\ f_1(x) &= q_2 f_2(x) - f_3(x), \\ &\cdots\cdots\cdots \\ f_{r-1}(x) &= q_r f_r(x) - f_{r+1}(x), \\ &\cdots\cdots\cdots \\ f_{n-2}(x) &= q_{n-1} f_{n-1}(x) - f_n(x). \end{aligned}\right\} \quad (1)$$

These equations involve the theory of the method of finding the greatest common measure; for it follows from the first equation that if $f(x)$ and $f_1(x)$ have a common factor, this must be a factor in $f_2(x)$; and from the second equation it follows, by like reasoning, that the same factor must occur in $f_3(x)$; and so on, till we come finally to the last remainder, which, when $f(x)$ and $f_1(x)$ have common factors, will be a polynomial consisting of these factors. In the present Article, where we suppose the given polynomial and its first derived to have no common factor, the last remainder $f_n(x)$ is numerical. It is essential for the proof of the theorem to observe also, that in the case now under consideration no two consecutive functions in the series can have a common factor; for if they had we could, by reasoning similar to the above, show from the equations that this factor must exist also in $f(x)$ and $f_1(x)$; and such, according to our hypothesis, is not here the case. In examining, therefore, what changes of sign can take place in the series during the passage of x from a to b, we may exclude the case of two consecutive functions vanishing for the same value of the variable; and the

different cases in which any change of sign can take place are the following :—

(1). when x passes through a root of the proposed equation $f(x) = 0$:

(2). when x passes through a value which causes one of the auxiliary functions $f_1, f_2, \ldots f_{n-1}$ to vanish :

(3). when x passes through a value which causes two or more of the series $f, f_1, \ldots f_{n-1}$ to vanish together ; no two of the vanishing functions, however, being consecutive.

(1). When x passes through a root of $f(x) = 0$, it follows from Art. 75 that one change of sign is lost, since immediately before the passage $f(x)$ and $f_1(x)$ have unlike signs, and immediately after the passage they have like signs.

(2). Suppose x to take a value a which satisfies the equation $f_r(x) = 0$. From the equation

$$f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x)$$

we have
$$f_{r-1}(a) = -f_{r+1}(a),$$

which proves that this value of x gives to $f_{r-1}(x)$ and $f_{r+1}(x)$ the same numerical value with different signs. In passing from a value a little less than a to one a little greater, we can suppose the interval so small that it contains no root of $f_{r-1}(x)$ or $f_{r+1}(x)$; hence, throughout the interval under consideration, these two functions retain their signs. If the sign of $f_r(x)$ does not change (as will happen in the exceptional case when the root a is repeated an even number of times) there is no alteration in the series of signs. In general the sign of $f_r(x)$ changes, but no variation of sign is either lost or gained thereby in the group of three; because, on account of the difference of signs of the two extremes $f_{r-1}(x)$ and $f_{r+1}(x)$, there will exist both before and after the passage one variation and one permanency of sign, whatever be the sign of the middle function. If, for example, before the passage the signs were $+ - -$; after the passage they are $+ + -$, *i. e.* a variation and a permanency are changed into a permanency and a variation ; but no variation of sign is lost or gained on the whole.

(3). Since the reasoning in the previous cases is founded on the relations of the function to those adjacent to it only; and since those relations remain unaltered in the present case, because no two adjacent functions vanish together, we conclude that if $f(x)$ is one of the vanishing functions, one change of sign is lost, and if not, no change is either lost or gained.

We have proved, therefore, that when x passes through a root of $f(x) = 0$ one change of sign is lost, and under no other circumstances is a change of sign either lost or gained. Hence the number of changes of sign lost during the variation of x from a to b is equal to the number of roots of the equation between a and b.*

Before proceeding to the case of equal roots, we add a few simple examples to illustrate the application of Sturm's theorem. It is convenient in practice to substitute first $-\infty$, 0, $+\infty$ in Sturm's functions, so as to obtain the whole number of negative and of positive roots. To separate the negative roots, the integers -1, -2, -3, &c., are to be substituted in succession till we reach the same series of signs as results from the substitution of $-\infty$; and to separate the positive roots we substitute 1, 2, 3, &c., till the signs furnished by $+\infty$ are reached.

Examples.

1. Find the number and situation of the real roots of the equation

$$f(x) \equiv x^3 - 2x - 5 = 0.$$

We find $\quad f_1(x) = 3x^2 - 2, \quad f_2(x) = 4x + 15, \quad f_3(x) = -643.$

Corresponding to the values $-\infty$, 0, $+\infty$ of x, we have

$$\begin{array}{lcccc} (-\infty) & - & + & - & -, \\ (0) & - & - & + & -, \\ (+\infty) & + & + & + & -. \end{array}$$

Hence there is only one real root, and it is positive.

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* The student often finds a difficulty in perceiving in what way a record is preserved in Sturm's series of the number of changes of sign lost, since the only loss takes place between the first two functions, $f(x)$ and $f_1(x)$. It may tend to remove this difficulty to observe, that as x increases from one root α of $f(x) = 0$ to a second β, although no alteration takes place in the *number* of changes of sign, the *distribution* of the signs among $f_1(x)$ and the following functions alters in such a way that the signs of $f(x)$ and $f_1(x)$, which were the same immediately after the passage of x through α, become again different before the passage through β.

Again, corresponding to values 1, 2, 3 of x, we have

$$\begin{array}{lcccc} (1) & - & + & + & -, \\ (2) & - & + & + & -, \\ (3) & + & + & + & -. \end{array}$$

The real root, therefore, lies between 2 and 3.

2. Find the number and situation of the real roots of the equation

$$x^3 - 7x + 7 = 0.$$

We easily obtain

$$f_1(x) = 3x^2 - 7,$$

$$f_2(x) = 2x - 3,$$

$$f_3(x) = 1;$$

whence

$$\begin{array}{lcccc} (-\infty) & - & + & - & +, \\ (0) & + & - & - & +, \\ (+\infty) & + & + & + & +. \end{array}$$

Hence all the roots are real: one negative, and two positive,

We have, further, the following results:—

$$\begin{array}{lcccc} (-4) & - & + & - & +, \\ (-3) & + & + & - & +, \\ (-2) & + & + & - & +, \\ (-1) & + & - & - & +, \\ (1) & + & - & - & +, \\ (2) & + & + & + & +. \end{array}$$

Here -4 and $+2$ give the same series of signs as $-\infty$ and $+\infty$; hence we stop at these. The negative root lies between -4 and -3; and the two positive roots between 1 and 2.

This example illustrates the superiority of Sturm's method over that of Fourier. The substitution of 1 and 2 in Fourier's functions gives, as can be immediately verified, the following series of signs:—

$$\begin{array}{lcccc} (1) & + & - & + & +, \\ (2) & + & + & + & +. \end{array}$$

From Fourier's theorem we are authorised to conclude only that there *cannot be more than* two roots between 1 and 2. From Sturm's we conclude that there *are* two roots between 1 and 2. If we have occasion to separate these two roots, we must, of course, make further substitutions in $f(x)$.

3. Find the number and situation of the real roots of the equation

$$x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$$

We obtain, removing the factor 2 from the derived,

$$f_1(x) = 2x^3 - 3x^2 - 3x + 5,$$
$$f_2(x) = 9x^2 - 27x + 11,$$
$$f_3(x) = -8x - 3,$$
$$f_4(x) = -1433.$$

[N.B.—In forming Sturm's functions it is allowable, as is evident from the equations (1), Art. 89, to introduce or suppress numerical factors just as in the process of finding the G. C. M.; taking care, however, that these are *positive*, so that the signs of the remainders are not thereby altered.]

We have the following series of signs:—

$(-\infty)$	+	−	+	+	−,
(0)	−	+	+	−	−,
$(+\infty)$	+	+	+	−	−.

Hence there are two real roots, one positive, and one negative, and two imaginary roots. To find the positions of the real roots, it is sufficient to substitute positive and negative integers successively in $f(x)$ alone, since there is only *one* positive and *one* negative root. We easily find in this way that the negative root lies between -2 and -3, and the positive root between 0 and 1.

90. **Sturm's Theorem. Equal Roots.** Let the operation for finding the greatest common measure of $f(x)$ and $f'(x)$ be performed, the signs of the successive remainders being changed as before. The last of Sturm's functions will not now be numerical, for since $f(x)$ and $f'(x)$ are here supposed to contain a common measure involving x, this will now be the last function arrived at by the process. Let the series of functions be:—

$$f(x), f_1(x), f_2(x), \ldots\ldots, f_r(x).$$

During the passage of x through any value except a multiple root of $f(x) = 0$, the conclusions of the last Article are still true with respect to the present series, since no value except such a root can cause any consecutive pair of the series to vanish. When x passes through a multiple root of $f(x) = 0$, there is, by the Cor., Art. 75, one change of sign lost between f and f_1; and we proceed to prove that no change of sign is lost or gained in the rest of the series, viz. $f_1, f_2, \ldots . f_r$. Suppose there exists an m-multiple root a of $f(x)$. It is evident from the equations (1) of Art. 89,

that $(x-a)^{m-1}$ is a factor in each of the functions $f_1, f_2, \ldots f_r$. Let the remaining factors in these functions be, respectively, $\phi_1, \phi_2, \ldots \phi_r$. By dividing each of the equations (1) by $(x-a)^{m-1}$, we get a series of equations which establish by the reasoning of the last Article that, owing to a passage through a, no change of sign is lost or gained in the series $\phi_1, \phi_2, \ldots \phi_r$. Neither, therefore, is any change lost or gained in the series $f_1, f_2, \ldots f_r$; for the effect of the factor $(x-a)^{m-1}$ in the passage of x from a value $a-h$ to a value $a+h$ is either to change the signs of all (when $m-1$ is odd) or of none (when $m-1$ is even) of the functions $\phi_1, \phi_2, \ldots \phi_r$; and changing the signs of all these functions cannot increase or diminish the number of variations.

We have therefore proved that when x passes through a multiple root of $f(x)=0$ one change of sign is lost between f and f_1, and none either lost or gained in any other part of the series. It remains true, of course, that when x passes through a single root of $f(x)=0$ a change of sign is lost as before. We may thus state the theorem as follows for the case of equal roots:—

The difference between the number of changes of sign when a and b are substituted in the series

$$f, f_1, f_2, \ldots f_r,$$

the last of these being the greatest common measure of f and f_1, is equal to the number of real roots between a and b, each multiple root counting only once.

Examples.

1. Find the nature of the roots of the equation

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = 0.$$

We easily obtain

$$f_1(x) = 4x^3 - 15x^2 + 18x - 7,$$

$$f_2(x) = x^2 - 2x + 1;$$

$f_2(x)$ divides $f_1(x)$ without remainder; hence in this case Sturm's series stops at $f_2(x)$, thus establishing the existence of equal roots.

To find the number of real roots of the equation, we substitute $-\infty$ and $+\infty$ for x in the series of functions f, f_1, f_2. The result is

$$\begin{array}{llll} (-\infty) & + & - & +, \\ (+\infty) & + & + & +. \end{array}$$

Hence the equation has only two real distinct roots; but one of these is a triple root, as is evident from the form of $f_2(x)$, which is equal to $(x-1)^2$.

2. Find the nature of the roots of the equation

$$x^4 - 6x^3 + 13x^2 - 12x + 4 = 0.$$

Here

$$f_1(x) = 4x^3 - 18x^2 + 26x - 12,$$

$$f_2(x) = x^2 - 3x + 2;$$

$f_2(x)$ is the last Sturmian function; so the equation has equal roots.

$$\begin{array}{llll} (-\infty) & + & - & +, \\ +\infty) & + & + & +. \end{array}$$

There are only two real distinct roots. In fact, since $f_2(x) \equiv (x-1)(x-2)$, each of the roots 1, 2 is a double root.

3. Find the nature of the roots of the equation

$$x^5 + 2x^4 + x^3 - x^2 - 2x - 1 = 0.$$

Here

$$f_1 = 5x^4 + 8x^3 + 3x^2 - 2x - 2,$$

$$f_2 = 2x^3 + 7x^2 + 12x + 7,$$

$$f_3 = -x^2 - 6x - 5,$$

$$f_4 = -x - 1,$$

$$f_5 = 0.$$

Since $f_5 = 0$, $x + 1$ is a common measure of f and f_1, and $f(x)$ has a double root -1. We have also

$$\begin{array}{llllll} (-\infty) & - & + & - & - & +, \\ (+\infty) & + & + & + & - & -. \end{array}$$

Hence there are two real distinct roots. The equation has, therefore, beside the double root, one other real root, and two imaginary roots.

4. Find the nature of the roots of the equation

$$x^6 - 7x^5 + 15x^4 - 40x^2 + 48x - 16 = 0.$$

Here

$$f_1(x) = 6x^5 - 35x^4 + 60x^3 - 80x + 48,$$

$$f_2(x) = 13x^4 - 84x^3 + 192x^2 - 176x + 48,$$

$$f_3(x) = x^3 - 6x^2 + 12x - 8 \equiv (x-2)^3.$$

Ans. There are three real distinct roots, one of them being quadruple.

91. **Application of Sturm's Theorem.**—In the case of equations of high degrees the calculation of Sturm's auxiliary functions becomes often very laborious. It is important, therefore, to pay attention to certain observations which tend somewhat to diminish this labour.

(1). In calculating the final remainder when it is numerical, since its sign is all we are concerned with, the labour of the last operation of division can be avoided by the consideration that the value of x which causes f_{n-1} to vanish must give opposite signs to f_{n-2} and f_n. It is in general possible to tell without any calculation what would be the sign of the result if the root of $f_{n-1}(x) = 0$ were substituted in $f_{n-2}(x)$. Thus in Ex. 3, Art. 89, if the value $-\frac{3}{8}$, which is the root of $f_3(x) = 0$, be substituted for x in $9x^2 - 27x + 11$, the result is evidently positive; hence the sign of $f_n(x)$ is $-$, and there is no occasion to calculate the value -1433 given for $f_n(x)$ in the example in question.

(2). When it is possible in any way to recognize that all the roots of any one of Sturm's functions are imaginary, we need not proceed to the calculation of any function beyond that one; for since such a function retains constantly the same sign for all values of the variable (Cor. Art. 12), no alteration in the number of changes of sign presented by it and the following functions can ever take place, so that the difference in the number of changes when two quantities a and b are substituted is independent of whatever variations of sign may exist in that part of the series which consists of the function in question and those following it. With a view to the application of this observation it is always well, when we arrive at the quadratic function ($ax^2 + bx + c$, suppose), to examine, in case the term containing x^2 and the absolute term have the same sign (otherwise the roots could not be imaginary), whether the condition $4ac > b^2$ is fulfilled; if so, we know that the roots are imaginary, and the calculation need not proceed farther.

Similar observations apply to the case where one of the functions is a perfect square, since such a function cannot change its sign for real values of x.

EXAMPLES.

1. Analyse the equation

$$x^4 + 3x^3 + 7x^2 + 10x + 1 = 0.$$

We find

$$f_2(x) = -29x^2 - 78x + 14,$$

$$f_3(x) = -1086x - 481,$$

$$f_4(x) = \quad -.$$

Here we see immediately that the value of x given by the equation $f_3(x) = 0$, which differs little from $-\frac{1}{2}$, makes $f_2(x)$ positive; hence $f_4(x)$ is negative. There are two real, and two imaginary roots. The real roots lie in the intervals $\{-2, -1\}$, $\{-1, 0\}$.

2. Analyse the equation

$$x^4 - 4x^3 - 3x + 23 = 0.$$

We find

$$f_2(x) = 12x^2 + 9x - 89,$$

$$f_3(x) = -491x + 1371,$$

$$f_4(x) = \quad -.$$

Here $f_3(x) = 0$ gives $x = \dfrac{1371}{491} > \dfrac{1371}{500} > 2{\cdot}74 > \dfrac{5}{2}$, and $x = \dfrac{5}{2}$ makes $f_2(x)$ positive; hence the root of $f_3(x)$ makes it positive also.

There are two real and two imaginary roots.

The real roots lie in the intervals $\{2, 3\}$, $\{3, 4\}$.

3. Analyse the equation

$$2x^4 - 13x^2 + 10x - 19 = 0.$$

Here

$$f_1(x) = 4x^3 - 13x + 5,$$

$$f_2(x) = 13x^2 - 15x + 38.$$

Since $4 \times 13 \times 38 > 15^2$, the roots of $f_2(x)$ are imaginary, and we proceed no farther with the calculation of Sturm's remainders.

Substituting $-\infty$, 0, $+\infty$, we obtain

$(-\infty)$	$+$	$-$	$+$,
(0)	$-$	$+$	$+$,
$(+\infty)$	$+$	$+$	$+$.

There are two real roots, one positive, the other negative.

4. Analyse the equation

$$f(x) \equiv x^5 + 2x^4 + x^3 - 4x^2 - 3x - 5 = 0.$$

Here

$$f_1(x) = 5x^4 + 8x^3 + 3x^2 - 8x - 3,$$

$$f_2(x) = 6x^3 + 66x^2 + 44x + 119,$$

$$f_3(x) = -116x^2 - 57x - 223.$$

Since $4 \times 116 \times 223 > 57^2$, we may stop the calculation here. We find, on substituting $-\infty$, 0, $+\infty$,

$$\begin{array}{lcccc} (-\infty) & - & + & - & -, \\ (0) & - & - & + & -, \\ (+\infty) & + & + & + & -. \end{array}$$

There are four imaginary roots, and one real positive root.

5. Find the number and situation of the real roots of the equation

$$x^4 - 2x^3 - 7x^2 + 10x + 10 = 0.$$

Ans. The roots are all real, and are situated in the intervals

$\{-3, -2\}$, $\{-1, 0\}$, and two between $\{2, 3\}$.

6. Analyse the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

It will be found that the calculation may cease with the quadratic remainder.

Ans. There is only one real root; in the interval $\{1, 2\}$.

7. Analyse the equation

$$x^3 + 11x^2 - 102x + 181 = 0.$$

We find
$$f_2(x) = 854x - 2751,$$
$$f_3(x) = 441.$$

In some examples, of which the present is an instance, it is not easy to tell immediately what sign the root of the penultimate function gives to the preceding function. We have here calculated $f_3(x)$, and it turns out to be a much smaller number than might have been expected from the magnitude of the coefficients in $f_2(x)$. In fact when the root of $f_2(x)$ is substituted in $f_1(x)$ the positive part is nearly equal to the negative part. This is always an indication that *two roots of the proposed equation are nearly equal.* There are in the present instance two positive roots between 3 and 4. Subdividing the intervals, we find the two roots still to lie between 3·2 and 3·3; so that they are very close together. We see here another illustration of the continuity which exists between real and imaginary roots. If $f_3(x)$ turned out to be zero, the roots would be actually equal. If it turned out to be a small negative number, the two roots would be imaginary.

8. Analyse the equation

$$x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0.$$

The quadratic function is found to have imaginary roots.

Ans. One real root between $\{0, 1\}$; four imaginary roots.

9. Analyse the equation

$$x^6 - 6x^5 - 30x^2 + 12x - 9 = 0.$$

We find

$$f_2(x) = 5x^4 + 20x^2 + 7;$$

and as this has plainly all imaginary roots, the calculation may stop here.

Ans. Two real roots; in the intervals $\{-2, -1\}$, $\{6, 7\}$.

10. Analyse the equation

$$2x^6 - 18x^5 + 60x^4 - 120x^3 - 30x^2 + 18x - 5 = 0.$$

We find

$$f_2(x) = 5x^4 + 220x^2 + 1;$$

and the calculation may stop.

Ans. Two real roots; in the intervals $\{-1, 0\}$, $\{5, 6\}$.

11. Examine how the roots of the equation

$$2x^3 + 15x^2 - 84x - 190 = 0$$

are situated in the several intervals between the numbers $-\infty$, -7, 6, $+\infty$.

Here

$$f_1(x) = x^2 + 5x - 14,$$

$$f_2(x) = 27x + 40,$$

$$f_3(x) = +.$$

The substitution of the above quantities gives

$(-\infty)$	$-$	$+$	$-$	$+$,
(-7)	$+$	0	$-$	$+$,
(6)	$+$	$+$	$+$	$+$,
$(+\infty)$	$+$	$+$	$+$	$+$.

Whenever, as in this example, any quantity makes one of the auxiliary functions vanish (here -7 satisfies $f_1(x) = 0$), the zero may be disregarded in counting the number of changes of sign in the corresponding row; for, since the signs on each side of it are different, no alteration in the number of changes of sign in the row could take place, whatever sign be supposed attached to the vanishing quantity.

The roots are all real. There is one root between $-\infty$ and -7; and two between -7 and 6.

12. Analyse the equation

$$3x^4 - 6x^2 - 8x - 3 = 0.$$

We find

$$f_1(x) = 3x^3 - 3x - 2,$$

$$f_2(x) = (x + 1)^2.$$

As $f_2(x)$ is a perfect square the calculation may cease.

Ans. Two real roots; in the intervals $\{-1, 0\}$, $\{1, 2\}$.

92. **Conditions for the Reality of the Roots of an Equation.**—The number of Sturm's functions, including $f(x)$, $f'(x)$ and the $n-1$ remainders, will in general be $n+1$. In certain cases, owing to the absence of terms in the proposed function, some of the remainders will be wanting. This can occur only when the proposed equation has imaginary roots; for it is plain that, in order to insure a loss of n changes of sign in the series of functions during the passage of x from $-\infty$ to $+\infty$ (namely, in order that the equation should have all its roots real), all the functions must be present. And, moreover, they must all take the same sign when $x = +\infty$; and alternating signs when $x = -\infty$. Since the leading term of an equation is always taken with a positive sign, we may state the condition for the reality of all the roots of any equation (supposed not to have equal roots) as follows:—*In order that all the roots of an equation of the n^{th} degree should be real, the leading coefficients of all Sturm's remainders, in number $n-1$, must be positive.*

Examples.

1. Find the condition that the roots of the equation

$$ax^2 + 2bx + c = 0$$

should be real and unequal.

Ans. $b^2 - ac > 0$.

2. Find the conditions that the roots of the cubic

$$z^3 + 3Hz + G = 0$$

should be all real and unequal.

When this cubic has its roots all real, it is evident that the general cubic from which it is derived (Art. 36) has also its roots all real; so that in investigating the conditions for the reality of the roots of a cubic in general, it is sufficient to discuss the form here written.

We find

$$f_1(z) = z^2 + H,$$

$$f_2(z) = -2Hz - G,$$

$$f_3(z) = -(G^2 + 4H^3).$$

[In calculating these, before dividing $f_1(z)$ by $f_2(z)$, multiply the former by the positive factor $2H^2$.]

Hence the required conditions are, H negative and $G^2 + 4H^3$ negative.

These can be expressed as one condition, viz., $G^2 + 4H^3$ negative, since this implies the former (cf. Art. 43).

3. Calculate Sturm's remainders for the biquadratic

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0.$$

We find

$$f_2(z) = -3Hz^2 - 3Gz - (a^2I - 3H^2),$$
$$f_3(z) = -(2HI - 3aJ)z - GI,$$
$$f_4(z) = I^3 - 27J^2.$$

These are obtained without much difficulty by aid of the identity of Art. 37. Before dividing f_1 by f_2, multiply by the positive factor $3H^2$; and when the remainder is found, remove the positive factor a^2. Before dividing f_2 by f_3, multiply by the positive factor $(2HI - 3aJ)^2$; and when the remainder is found, remove the positive factor a^2H^2.

93. **Conditions for the Reality of the Roots of a Biquadratic.**—In order to arrive at criteria of the nature of the roots of the general algebraic equation of the fourth degree by Sturm's method, it is sufficient to consider the equation of Ex. 3 of the preceding Article. By aid of the forms of the leading coefficients of Sturm's remainders there calculated, we can write down the *conditions that all the roots of a biquadratic should be real and unequal* in the form

H *negative*, $2HI - 3aJ$ *negative*, $I^3 - 27J^2$ *positive.*

It will be observed that the second of these conditions is different in form from the corresponding condition of Art. 68. To show the equivalence of the two forms it is necessary to prove that when H is negative and Δ positive, the further condition $2HI - 3aJ$ negative implies the condition $a^2I - 12H^2$ negative, and conversely. From the identity of Art. 37, written in the form $-H(a^2I - 12H^2) \equiv a^2(2HI - 3aJ) - 3G^2$, it readily appears that when H and $2HI - 3aJ$ are negative $a^2I - 12H^2$ is necessarily negative. And to prove the converse we observe that when aJ is positive $2HI - 3aJ$ is negative, since I is positive on account of the condition Δ positive; and when aJ is negative $2HI - 3aJ$ is still negative, since the negative part $2HI$ exceeds the positive part $-3aJ$, as may be readily shown by the aid of the inequalities $12H^2 > a^2I$ and $I^3 > 27J^2$.

The student will have no difficulty in verifying, by means of Sturm's functions, the remaining conclusions arrived at in the different cases of Art. 68.

EXAMPLES.

1. Apply Budan's method to separate the roots of the equation

$$x^4 - 16x^3 + 69x^2 - 70x - 42 = 0.$$

Ans. Roots in intervals $\{-1, 0\}$, $\{2, 3\}$, $\{4, 5\}$, $\{9, 10\}$.

2. Apply Sturm's theorem to the analysis of the equation

$$x^4 - 4x^3 + 7x^2 - 6x - 4 = 0.$$

In analysing a biquadratic of this nature which has plainly two real roots, when a Sturmian remainder is reached whose leading coefficient is negative, the calculation may cease, since the other pair of roots must then be imaginary, and the positions of the real roots can be readily found by substitution in the given equation.

Ans. Two roots imaginary; real roots in intervals $\{-1, 0\}$, $\{2, 3\}$.

3. Analyse in a similar manner the equation

$$x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$$

Ans. Two roots imaginary; real roots in intervals $\{-1, 0\}$, $\{3, 4\}$.

4. Apply Sturm's theorem to the analysis of the equation

$$x^4 + 3x^3 - x^2 - 3x + 11 = 0.$$

Ans. Roots all imaginary.

5. Find by Sturm's method the number and positions of the real roots of the equation

$$x^5 - 10x^3 + 6x + 1 = 0.$$

Ans. Roots all real; one in the interval $\{-4, -3\}$; two in the interval $\{-1, 0\}$; and positive roots in the intervals $\{0, 1\}$, $\{3, 4\}$.

6. Calculate Sturm's functions for the following equation, and show that all the roots are real:—

$$x^5 - 5x^4 + 5x^3 + 5x^2 - 5x - 1 = 0.$$

7. Calculate Sturm's functions for the following equation, and show that four roots are imaginary:—

$$3x^5 + 5x^3 + 2 = 0.$$

This and the preceding example are instances in which, as the student will easily see, there is a factor common to two of Sturm's remainders which are not consecutive.

8. Calculate Sturm's functions for the following equation, and verify the conclusions of Ex. 23, p. 104, with regard to the character of the roots:—

$$x^5 - 5px^3 + 5p^2x + 2q = 0.$$

9. Prove that when all Sturm's functions are present, the number of changes of sign among the coefficients of the leading terms is equal to the number of pairs of imaginary roots of the equation.

10. If the signs of the leading coefficients of the first two of Sturm's remainders for a quintic be $-+$, prove that the number of real roots is determined.

Ans. One real root only.

11. If H and J are both positive, prove that all the roots of the biquadratic are imaginary; and that under the same conditions the quintic written with binomial coefficients has only one real root. Mr. M. Roberts, *Dublin Exam. Papers*, 1862.

12. Calculate the first two of Sturm's remainders for a quintic wanting the second term, viz.

$$f(x) \equiv x^5 + ax^3 + bx^2 + cx + d = 0.$$

Ans.
$$R_1 \equiv -2ax^3 - 3bx^2 - 4cx - 5d,$$

$$R_2 \equiv Ax^2 + Bx + C,$$

where

$$A \equiv 40ac - 12a^3 - 45b^2, \quad B \equiv 50ad - 8\, a^2 b - 60bc, \quad C \equiv -4a^2c - 75bd.$$

Retaining this notation, it is easy to calculate the coefficients D, E of the third remainder $R_3 \equiv Dx + E$ in terms of a, b, c, d, A, B, C; and, finally, R_4 in terms of A, B, C, D, E.

13. Remove the second term from the general quintic written with binomial coefficients, and prove that the leading coefficients of the first two of Sturm's remainders for the resulting equation are

$$-H, \quad -5HI + 9a_0J.$$

14. Prove that, if c has any value except unity, the equation

$$c^2x^4 - 2c^2x^3 + 2x - 1 = 0$$

has a pair of imaginary roots.

15. Prove that the roots of the equation

$$x^3 - (a^2 + b^2 + c^2)\,x - 2abc = 0$$

are all real, and solve it when two of the quantities a, b, c become equal.

16. Prove that when the biquadratic

$$f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

has a triple factor, it may be expressed in the form

$$a^3f(x) \equiv \{ax + b + \sqrt{-H}\}^3\,\{ax + b - 3\sqrt{-H}\}.$$

17. Verify by means of Sturm's remainders the conditions which must be fulfilled when the biquadratic of the previous example is a perfect square, and prove in that case

$$a^3f(x) \equiv \{(ax + b)^2 + 3H\}^2.$$

18. If an equation of any degree, arranged according to powers of x, have three consecutive terms in geometric progression, prove that its roots cannot be all real.

These three terms must be of the form $kx^r + k\alpha x^{r-1} + k\alpha^2 x^{r-2}$. Let the equation be multiplied by $x - \alpha$. The resulting equation will have two consecutive terms absent, and must therefore have at least two imaginary roots; but all the roots of this equation except α are roots of the given equation.

19. If an equation have four consecutive coefficients in arithmetic progression, prove that its roots cannot be all real.

This can be reduced to the preceding example. Writing down four terms of the proper form, and multiplying by $x - 1$, it readily appears that the resulting equation has three consecutive terms in geometric progression.

20. If all the roots of any equation $f(x) = 0$ are real, prove that all the roots of every one of Sturm's auxiliary functions are also real.

This can be established by reasoning similar to that of Art. 89. Consider the k^{th} remainder R_k, and let its degree be m. This and the m functions which follow constitute a series, of which no adjacent two can vanish together. When $x = -\infty$, their signs are alternately positive and negative, and when $x = +\infty$, they are all positive. There are, therefore, m changes of sign to be lost as x passes from $-\infty$ to $+\infty$; and no change of sign can be lost except on the passage through a root of $R_k = 0$, which equation must consequently have m real roots.

Since a value of x which causes any of the functions to vanish gives opposite signs to the two adjacent functions, it is easily inferred that any equation of the series is a limiting equation with regard to the function which precedes it.

21. If the real roots of any one, $f_m(x)$, of the Sturmian auxiliary functions be known, prove that the number and positions of the roots of the original equation may be determined without the aid of the functions below $f_m(x)$.

Let the real roots, in order of magnitude, of $f_m(x) = 0$ be $\alpha, \beta, \ldots \eta, \theta$; the remaining roots being imaginary. As x varies from $-\infty$ to a value a little less than θ, the function $f_m(x)$ cannot change its sign; and therefore in examining the roots of $f(x) = 0$ which lie between these limits, the Sturmians which follow $f_m(x)$ may be disregarded. The same holds true as x passes from a value a little greater than θ to one a little less than η; and similarly for the remaining intervals. If therefore we examine separately the intervals $\{-\infty, \theta\}$, $\{\theta, \eta\}$, $\ldots \{\beta, \alpha\}$, $\{\alpha, +\infty\}$, the number of roots of the original equation which lie in each of these regions can be determined without the aid of the lower Sturmian functions.

22. If any one of Sturm's auxiliary functions has imaginary roots, the original equation has at least an equal number of imaginary roots. (Mr. F. Purser.)

This can be inferred from the preceding example by examining the greatest possible number of changes which can be lost in the series terminating with $f_m(x)$, during the passage of x from $-\infty$ to $+\infty$; remembering that, so far as the limited series is concerned, a change of sign may be gained on the passage through each real root of $f_m(x) = 0$.

CHAPTER X.

SOLUTION OF NUMERICAL EQUATIONS.

94. **Algebraical and Numerical Equations.**—There is an essential distinction between the solutions of algebraical and numerical equations. In the former the result is a general formula of a purely symbolical character, which, being the general expression for a root, must represent all the roots indifferently. It must be such that, when for the functions of the coefficients involved in it the corresponding symmetric functions of the roots are substituted, the operations represented by the radical signs $\sqrt{}$, $\sqrt[3]{}$ become practicable; and when the square and cube roots of these symmetric functions are extracted, the whole expression in terms of the roots will reduce down to one root: the different roots resulting from the different combinations $\pm\sqrt{}$ of square roots, and $\sqrt[3]{}$, $\omega\sqrt[3]{}$, $\omega^2\sqrt[3]{}$ of cube roots. For a simple illustration of what is here stated we refer to the case of the quadratic in Art. 55. In Articles 59 and 66 we have similar illustrations for the cubic and biquadratic. It is to be observed also that the formula which represents the root of an algebraic equation holds good even when the coefficients are imaginary quantities.

In the case of numerical equations the roots are determined separately by the methods we are about to explain; and, before attempting the approximation to any individual root, it is in general necessary that it should be situated in a known interval which contains no other real root.

The real roots of numerical equations may be either commensurable or incommensurable; the former class including integers, fractions, and terminating or repeating decimals which

are reducible to fractions; the latter consisting of interminable decimals. The roots of the former class can be found exactly, and those of the latter approximated to with any degree of accuracy, by the methods we are about to explain.

We shall commence by establishing a theorem which reduces the determination of the former class of roots to that of *integral roots* alone.

95. **Theorem.**—*An equation in which the coefficient of the first term is unity, and the coefficients of the other terms whole numbers, cannot have a commensurable root which is not a whole number.*

For, if possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0;$$

we have then

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \ldots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0;$$

from which, multiplying by b^{n-1}, we obtain

$$-\frac{a^n}{b} = p_1a^{n-1} + p_2a^{n-2}b + \ldots + p_{n-1}ab^{n-2} + p_nb^{n-1}.$$

Now a^n is not divisible by b, and each term on the right-hand side of the equation is an integer. We have, therefore, a fraction in its lowest terms equal to an integer, which is impossible. Hence $\frac{a}{b}$ cannot be a root of the equation. The real roots of the equation, therefore, are either integers or incommensurable quantities.

Every equation whose coefficients are finite numbers, fractional or not, can be reduced to the form in which the coefficient of the first term is unity and those of the other terms whole numbers (Art. 31); so that in this way, by the aid of a simple transformation, the determination of the commensurable roots in general can be reduced to that of integral roots.

We proceed to explain Newton's process, called the Method of Divisors, of obtaining the integral roots of an equation whose coefficients are all integers.

96. **Newton's Method of Divisors.**—Suppose h to be an integral root of the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0. \tag{1}$$

Let the quotient, when the polynomial is divided by $x - h$, be

$$b_0 x^{n-1} + b_1 x^{n-2} + \ldots + b_{n-2} x + b_{n-1},$$

in which b_0, b_1, &c., are plainly all integers.

Proceeding as in Art. 8, we obtain the following equations:—

$$a_0 = b_0, \quad a_1 = b_1 - hb_0, \quad a_2 = b_2 - hb_1, \ldots$$

$$a_{n-2} = b_{n-2} - hb_{n-3}, \quad a_{n-1} = b_{n-1} - hb_{n-2}, \quad a_n = -hb_{n-1}.$$

The last of these equations proves that a_n is divisible by h, the quotient being $-b_{n-1}$. The second last, which is the same as

$$a_{n-1} + \frac{a_n}{h} = -hb_{n-2},$$

proves that the sum of the quotient thus obtained and the second last coefficient is again divisible by h, the quotient being $-b_{n-2}$; and so on.

Continuing the process, the last quotient obtained in this way will be $-b_0$, which is equal to $-a_0$.

If we perform the process here indicated with all the divisors of a_n which lie within the limits of the roots, those which satisfy the above conditions, giving integral quotients at each step, and a final quotient equal to $-a_0$, are roots of the proposed equation. Those which at any stage of the process give a fractional quotient are to be rejected.

When the coefficient $a_0 = 1$, we know by the theorem of the last Article that the integral roots determined in this way are all the commensurable roots of the proposed equation. If a_0 be

not = 1, the process will still give the integral roots of the equation as it stands; but to be sure of determining in this way all the commensurable roots, the equation must be first transformed to one which shall have the coefficient of the highest term equal to unity.

97. **Application of the Method of Divisors.**—With a view to the most convenient mode of applying the Method of Divisors, we write the series of operations as follows, in a manner analogous to Art. 8:—

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} \;.\;.\;.\;.\; a_2 & a_1 & a_0 \\ & -b_{n-1} & -b_{n-2} \qquad -b_2 & -b_1 & -b_0 \\ \hline & -hb_{n-2}, & -hb_{n-3} \qquad -hb_1 & -hb_0 & 0 \end{array}$$

The first figure in the second line $(-b_{n-1})$ is obtained by dividing a_n by h. This is to be added to a_{n-1} to obtain the first figure in the third line $(-hb_{n-2})$. This is to be divided by h to obtain the second figure in the second line $(-b_{n-2})$; this to be added to a_{n-2}; and so on. If h be a root, the last figure in the second line thus obtained will be $-a_0$.

When we succeed in proving in this manner that any integer h is a root, the next operation with any divisor may be performed, not on the original coefficients a_n, a_{n-1}, ... , but on those of the second line with their signs changed, for these are the coefficients of the quotient when the original polynomial is divided by $x-h$. When any divisor gives at any stage a fractional result it is to be rejected at once, and the operation so far as it is concerned stopped.

The numbers 1 and -1, which are always of course integral divisors of a_n, need not be included in the number of trial divisors. It is more convenient before applying the Method of Divisors to determine by direct substitution whether either of these numbers is a root.

Examples.

1. Find the integral roots of the equation

$$x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$$

By grouping the terms (see Art. 79) we observe without difficulty that all the roots lie between -5 and $+5$. The following divisors are possible roots :—

$$-4, \quad -3, \quad -2, \quad 2, \quad 3, \quad 4.$$

We commence with 4 :—

$$\begin{array}{ccccc} -24 & 38 & -13 & -2 & 1 \\ & -6 & 8 & & \\ \hline & 32 & -5 & & \end{array}$$

The operation stops here, for since -5 is not divisible by 4, 4 cannot be a root. We proceed then with the number 3 :

$$\begin{array}{ccccc} -24 & 38 & -13 & -2 & 1 \\ & -8 & 10 & -1 & -1 \\ \hline & 30 & -3 & -3 & 0\,; \end{array}$$

hence 3 is a root ; and in proceeding with the next integer, 2, we make use, as above explained, of the coefficients of the second line with signs changed :

$$\begin{array}{cccc} 8 & -10 & 1 & 1 \\ & 4 & -3 & -1 \\ \hline & -6 & -2 & 0\,; \end{array}$$

hence 2 also is a root ; and we proceed with -2 :

$$\begin{array}{ccc} -4 & 3 & 1 \\ & 2 & \\ \hline & 5\,; & \end{array}$$

hence -2 is not a root, for it does not divide 5. -3 is plainly not a root, for it does not divide -4.

[We might at once have struck out -3 as not being a divisor of the absolute term 8 of the reduced polynomial. This remark will often be of use in diminishing the number of divisors.]

We proceed now to the last divisor, -4.

-4	3	1
	1	-1
	4	0

Thus -4 is a root.

The equation has, therefore, the integral roots 3, 2, -4; and the last stage of the operation shows that when the original polynomial is divided by the binomials, $x-3$, $x-2$, $x+4$, the result is $x-1$; so that 1 is also a root. Hence the original polynomial is equivalent to

$$(x-1)(x-2)(x-3)(x+4).$$

2. Find the integral roots of

$$3x^4 - 23x^3 + 35x^2 + 31x - 30 = 0.$$

The roots lie between -2 and 8; hence we have only to test the divisors 2, 3, 5, 6.

We find immediately that 6 is not a root.

For 5 we have

-30	31	35	-23	3
	-6	5	8	-3
	25	40	-15	0;

hence 5 is a root. For 3 we have

6	-5	-8	3
	2	-1	-3
	-3	-9	0;

hence 3 is a root; and we easily find that 2 is not a root.

The quotient, when the original polynomial is divided by $(x-5)(x-3)$, is, from the last operation,

$$3x^2 + x - 2:$$

of this 1 is not a root, and -1 is a root. Hence all the integral roots of the proposed equation are -1, 3, 5.

The other root of the equation is $\frac{2}{3}$. It is a commensurable root; but, not being integral, is not given in the above operation.

3. Find all the roots of the equation

$$x^4 + x^3 - 2x^2 + 4x - 24 = 0.$$

Limits of the roots are -4, 3.

Ans. Roots -3, 2, $\pm 2\sqrt{-1}$.

4. Find all the roots of the equation

$$x^4 - 2x^3 - 19x^2 + 68x - 60 = 0.$$

The roots lie between -6 and 6.

We find that $2, 3, -5$ are roots, and that the factor left after the final division is $x - 2$; hence 2 is a double root. The polynomial is therefore equivalent to

$$(x-2)^2 (x-3)(x+5).$$

In Art. 99 the case of multiple roots will be further considered.

98. **Method of Limiting the Number of Divisors.**— It is possible of course to determine by direct substitution whether any of the divisors of a_n are roots of the proposed equation; but Newton's method has the advantage, as the above examples show, that some of the divisors are rejected after very little labour. It has a further advantage which will now be explained. When the number of divisors of a_n within the limits of the roots is large, it is important to be able, before proceeding with the application of the method in detail, to diminish the number of these divisors which need be tested. This can be done as follows:—

If h is an integral root of $f(x) = 0$, $f(x)$ is divisible by $x - h$, and the coefficients of the quotient are integers, as was above explained. If therefore we assign to x any integral value, the quotient of the corresponding value of $f(x)$ by the corresponding value of $x - h$ must be an integer. We take, for convenience, the simplest integers 1 and -1; and, before testing any divisor h, we subject it to the condition that $f(1)$ must be divisible by $1 - h$ (or, changing the sign, by $h - 1$); and that $f(-1)$ must be divisible by $-1 - h$ (or, changing the sign, by $1 + h$).

In applying this observation it will be found convenient to calculate $f(1)$ and $f(-1)$ in the first instance: if either of these vanishes, the corresponding integer is a root, and we proceed with the operation on the reduced polynomial whose coefficients have been ascertained in the process of finding the result of substituting the integer in question.

EXAMPLES.

1. $$x^5 - 23x^4 + 160x^3 - 281x^2 - 257x - 440 = 0.$$

The roots lie between -1 and 24.
We have the following divisors:—

2, 4, 5, 8, 10, 11, 20, 22.

We easily find
$$f(1) = -840, \quad \text{and} \quad f(-1) = -648.$$

We therefore exclude all the above divisors, which, when diminished by 1, do not divide 840; and which, when increased by 1, do not divide 648. The first condition excludes 10 and 20, and the second 4 and 22. Applying the Method of Divisors to the remaining integers 2, 5, 8, 11, we find that 5, 8, and 11 are roots, and that the resulting quotient is $x^2 + x + 1$. Hence the given polynomial is equivalent to
$$(x-5)(x-8)(x-11)(x^2+x+1).$$

2. $$x^5 - 29x^4 - 31x^3 + 31x^2 - 32x + 60 = 0.$$

The roots lie between -3 and 32.

Divisors: -2, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30.

$f(1) = 0$; so 1 is a root.
$f(-1) = 124$; and the above condition excludes all the divisors except -2, 3, 30.
We easily find that -2 and 30 are roots, and that the final quotient is $x^2 + 1$. The given polynomial is equivalent to $(x-1)(x-30)(x+2)(x^2+1)$.

99. **Determination of Multiple Roots.**—The Method of Divisors determines multiple roots when they are commensurable. In applying the method, when any divisor of a_n which is found to be a root is a divisor of the absolute term of the reduced polynomial, we must proceed to try whether it is also a root of the latter, in which case it will be a double root of the proposed equation. If it be found to be a root of the next reduced polynomial, it will be a triple root of the proposed; and so on. Whenever in an equation of any degree there exists only *one* multiple root, r times repeated, it can be found in this way; for the common measure of $f(x)$ and $f'(x)$ will then be of the form $(x-a)^{r-1}$, and the coefficients of this could not be commensurable if a were incommensurable.

Multiple roots of equations of the third, fourth, and fifth degrees can be completely determined without the use of the process of finding the greatest common measure, as will appear from the following observations:—

(1). *The Cubic.*—In this case multiple roots must be commensurable, since the degree is not high enough to allow of two distinct roots being repeated.

(2). *The Biquadratic.*—In this case either the multiple roots are commensurable or the function is a perfect square. For the only form of biquadratic which admits of two distinct roots being repeated is

$$(x-\alpha)^2(x-\beta)^2,$$

i.e. the square of a quadratic. The roots of the quadratic may be incommensurable. If we find, therefore, that a biquadratic has no commensurable roots, we must try whether it is a perfect square in order to determine further whether it has equal incommensurable roots.

(3). *The Quintic.*—In this case, either the multiple roots are commensurable, or the function consists of a linear commensurable factor multiplied by the square of a quadratic factor. For, in order that two distinct roots may be repeated, the function must take one or other of the forms

$$(x-\alpha)^2(x-\beta)^2(x-\gamma), \quad (x-\alpha)^2(x-\beta)^3.$$

In the latter case the roots cannot be incommensurable; but the former may correspond to the case of a commensurable factor multiplied by the square of a quadratic whose roots are incommensurable. If then a quintic be found to have no commensurable roots, it can have no multiple roots. If it be found to have one commensurable root only, we must examine whether the remaining factor is a perfect square. If it have more than one commensurable root, the multiple roots will be found among the commensurable roots.

Examples.

1. Find all the commensurable roots of

$$2x^3 - 31x^2 + 112x + 64 = 0.$$

The roots lie between the limits -1, 16. The divisors are 2, 4, 8.

$$\begin{array}{cccc} 64 & 112 & -31 & 2 \\ & 8 & 15 & -2 \\ \hline & 120 & -16 & 0; \end{array}$$

8 is therefore a root. Proceed now with the reduced equation:

$$\begin{array}{ccc} -8 & -15 & 2 \\ & -1 & -2 \\ \hline & -16 & 0 \end{array}$$

8 is a root again, and the remaining factor is $2x + 1$.

Ans. $f(x) \equiv (2x + 1)(x - 8)^2$.

2. Find the commensurable and multiple roots of

$$x^4 - x^3 - 30x^2 - 76x - 56 = 0.$$

The roots lie between the limits -6, 12. (Apply method of Ex. 10, Art. 80).

Ans. $f(x) \equiv (x + 2)^3(x - 7)$.

3. Find the commensurable and multiple roots of

$$9x^4 - 12x^3 - 71x^2 - 40x + 16 = 0.$$

The roots lie between the limits -2, 5.

The equation as it stands is found to have no integral root; but it may still have a commensurable root. To test this we multiply the roots by 3 in order to get rid of the coefficient of x^4. We find then

$$x^4 - 4x^3 - 71x^2 - 120x + 144 = 0.$$

Limits: -6, 15.

We find -4 to be a double root of this, and the function to be equivalent to $(x^2 - 12x + 9)(x + 4)^2$. The original equation is therefore identical with the following:—

$$(x^2 - 4x + 1)(3x + 4)^2 = 0.$$

4. Find the commensurable and multiple roots of

$$x^4 + 12x^3 + 32x^2 - 24x + 4 = 0.$$

The roots lie between -12 and 1. The only divisors to be tested are, therefore, $-4, -2, -1$. We find that the equation has no commensurable root. We proceed to try whether the given function is a perfect square. This can be done by extracting the square root, or by applying the conditions of Ex. 3, p. 125. We find that it is the square of $x^2 + 6x - 2$ (cf. Ex. 1, p. 163). Hence the given equation has two pairs of equal roots, both incommensurable.

5. Find the commensurable and multiple roots of

$$f(x) \equiv x^5 - x^4 - 12x^3 + 8x^2 + 28x + 12 = 0.$$

The limits of the roots are $-4, 4$.

We find that -3 is a root, and that the reduced equation is

$$x^4 - 4x^3 + 8x + 4 = 0,$$

and that there is no other commensurable root.

The only case of possible occurrence of multiple roots is, therefore, when this latter function is a perfect square. It is found to be a perfect square, and we have

$$f(x) \equiv (x^2 - 2x - 2)^2 (x + 3).$$

6. Find the commensurable and multiple roots of

$$f(x) \equiv x^5 - 8x^4 + 22x^3 - 26x^2 + 21x - 18 = 0.$$

Ans. $f(x) \equiv (x^2 + 1)(x - 2)(x - 3)^2$.

7. The following equation has only two different roots: find them :—

$$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0.$$

In general it is obvious that if an integral root h occurs twice, the last coefficient must contain h^2 as a factor, and the second last h; if the root occurs three times, h^3 must be a factor of the last, h^2 of the second last, and h of the third last coefficient. The last coefficient here $= 2^2 \cdot 3^3$. Hence, if neither -1 nor 1 is a root, the required roots must be 2 and 3. That these are the roots is easily verified.

8. The equation

$$800x^4 - 102x^2 - x + 3 = 0$$

has equal roots: find all the roots.

In this example it is convenient to change the roots into their reciprocals before applying the Method of Divisors.

Ans. $f(x) \equiv (10x - 3)(5x - 1)(4x + 1)^2$.

100. **Newton's Method of Approximation.**—Having shown how the commensurable roots of equations may be obtained, we proceed to give an account of certain methods of obtaining approximate values of the incommensurable roots. The method of approximation, commonly ascribed to Newton,* which forms the subject of the present Article, is valuable as

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* See Note B at the end of the volume.

being applicable to numerical equations involving transcendental as well as those involving algebraical functions only. Although when applied to the latter class of functions Newton's method is, for practical purposes, inferior in form to Horner's, which will be explained in the following Articles, yet in principle both methods are to a great extent identical.

In all methods of approximation the root we are seeking is supposed to be separated from the other roots, and to be situated in a known interval between close limits.

Let $f(x) = 0$ be a given equation, and suppose a value a to be known, differing by a small quantity h from a root of the equation. We have then, since $a + h$ is a root of the equation, $f(a+h) = 0$; or

$$f(a) + f'(a)\,h + \frac{f''(a)}{1 \cdot 2}h^2 + \ldots\ldots = 0.$$

Neglecting now, since h is small, all powers of h higher than the first, we have

$$f(a) + f'(a)\,h = 0,$$

giving, as a first approximation to the root, the value

$$a - \frac{f(a)}{f'(a)}.$$

Representing this value by b, and applying the same process a second time, we find as a closer approximation

$$b - \frac{f(b)}{f'(b)}.$$

By repeating this process the approximation can be carried to any degree of accuracy required.

EXAMPLE.

Find an approximate value of the positive root of the equation

$$x^3 - 2x - 5 = 0.$$

The root lies between 2 and 3 (Ex. 1, Art. 89). Narrowing the limits, the root is found to lie between 2 and 2·2. We take 2·1 as the quantity represented by a. It cannot differ from the true value $a + h$ of the root by more than 0·1. We find easily

$$\frac{f(a)}{f'(a)} = \frac{f(2 \cdot 1)}{f'(2 \cdot 1)} = \frac{\cdot 061}{11.23} = 0 \cdot 00543.$$

A first approximation is, therefore,

$$2{\cdot}1 - 0{\cdot}00543 = 2{\cdot}0946.$$

Taking this as b, and calculating the fraction $\frac{f(b)}{f'(b)}$, we obtain

$$b - \frac{f(b)}{f'(b)} = 2{\cdot}09455148$$

for a second approximation; and so on.

The approximation in Newton's method is, in general, rapid. When, however, the root we are seeking is accompanied by another nearly equal to it, the fraction $\frac{f(a)}{f'(a)}$ is not necessarily small, since the value of either of the nearly equal roots reduces $f'(x)$ to a small quantity. A case of this kind requires special precautions. We do not enter into any further discussion of the method, since for practical purposes it may be regarded as entirely superseded by Horner's method, which will now be explained.

101. **Horner's Method of Solving Numerical Equations.**—By this method both the commensurable and incommensurable roots can be obtained. The root is evolved figure by figure: first the integral part (if any), and then the decimal part, till the root terminates if it be commensurable, or to any number of places required if it be incommensurable. The process is similar to the known processes of extraction of the square and cube root, which are, indeed, only particular cases of the general solution by the present method of quadratic and cubic equations.

The main principle involved in Horner's method is the successive diminution of the roots of the given equation by known quantities, in the manner explained in Art. 33. The great advantage of the method is, that the successive transformations are exhibited in a compact arithmetical form, and the root obtained by one continuous process correct to any number of places of decimals required.

This principle of the diminution of the roots will be illustrated in the present Article by some simple examples. In the

following Articles we shall proceed to certain considerations which tend to facilitate the practical application of the method.

Examples.

1. Find the positive root of the equation

$$2x^3 - 85x^2 - 85x - 87 = 0.$$

The first step, when any numerical equation is proposed for solution, is to find the *first figure* of the root. This can usually be done by a few trials: although in certain cases the methods of separation of the roots explained in Chap. IX. may have to be employed. In the present example there can be only one positive root; and it is found by trial to lie between 40 and 50. Thus the first figure of the root is 4. We now diminish the roots by 40. The transformed equation will have one root between 0 and 10. It is found by trial to lie between 3 and 4. We now diminish the roots of the transformed equation by 3; so that the roots of the proposed equation will be diminished by 43. The second transformed equation will have one root between 0 and 1. On diminishing the roots of this latter equation by ·5, we find that its absolute term is reduced to zero, *i.e.* the diminution of the roots of the proposed equation by 43·5 reduces its absolute term to zero. We conclude that 43·5 is a root of the given equation. The series of arithmetical operations is represented as follows:—

2	−85	−85	−87	(43·5
	80	−200	−11400	
	−5	−285	**−11487**	
	80	3000	9594	
	75	**2715**	**−1893**	
	80	483	1893	
	155	3198	0	
	6	501		
	161	**3699**		
	6	87		
	167	3786		
	6			
	173			
	1			
	174			

The broken lines mark the conclusion of each transformation, and the figures in dark type are the coefficients of the successive transformed equations (see Art. 33). Thus

$$2x^3 + 155x^2 + 2715x - 11487 = 0$$

is the equation whose roots are each less by 40 than the roots of the given equation, and whose positive root is found to lie between 3 and 4. If the second transformed equation had not an exact root ·5; but one, we shall suppose, between ·5 and ·6, the first three figures of the root of the proposed equation would be 43·5; and to find the next figure we should proceed to a further transformation, diminishing the roots by ·5; and so on.

2. Find the positive root of the equation

$$4x^3 - 13x^2 - 31x - 275 = 0.$$

We first write down the arithmetical work, and proceed to make certain observations on it:—

4	− 13	− 31	− 275	(6·25
	24	66	210	
	11	35	**− 65**	
	24	210	51·392	
	35	**245**	**− 13·608**	
	24	11·96	13·608	
	59	256·96	0	
	·8	12·12		
	59·8	**269·08**		
	·8	3·08		
	60·6	272·16		
	·8			
	61·4			
	·2			
	61·6			

We find by trial that the proposed equation has its positive root between 6 and 7. The first figure of the root is, therefore, 6. Diminish the roots by 6. The equation

$$x^3 + 59x^2 + 245x - 65 = 0$$

has, therefore, a root between 0 and 1. It is found by trial to lie between ·2 and ·3. The first two figures of the root of the proposed are therefore 6·2. Diminish the roots again by ·2. The transformed equation is found to have the root ·05. Hence 6·25 is a root of the proposed equation.

It is convenient in practice to avoid the use of the decimal points. This can easily be effected as follows:—When the decimal part of the root (suppose ·*abc*...) is about to appear, multiply the roots of the corresponding transformed equation by 10, *i.e.* annex one zero to the right of the figure in the first column, two to the right of the figure in the second column, three to the right of that in the third; and so on, if there be more columns (as there will of course be in equations of a degree higher than the third). The root of the transformed equation is then, not ·*abc*..., but *a*·*bc*... Diminish the roots by *a*. The transformed equation has a root ·*bc*... Multiply the roots of this equation again by 10. The root becomes *b*·*c*..., and

the process is continued as before. To illustrate this we repeat the above operation, omitting the decimal points. In all subsequent examples this simplification will be adopted:—

4	− 13	− 31	− 275	(6·25
	24	66	210	
	11	35	− **65000**	
	24	210	51392	
	35	**24500**	− **13608000**	
	24	1196	13608000	
	590	25696	0	
	8	1212		
	598	**2690800**		
	8	30800		
	606	2721600		
	8			
	6140			
	20			
	6160			

3. Find the positive root of the equation

$$20x^3 - 121x^2 - 121x - 141 = 0.$$

The root is easily found to lie between 7 and 8. It is, therefore, of the form $7\,.\,ab\ldots$ When the roots are diminished by 7, and multiplied by 10, the resulting equation is

$$20x^3 + 2990x^2 + 112500x - 57000 = 0.$$

The positive root of this is $a\,.\,b\ldots$; and as the root plainly lies between 0 and 1, we have $a = 0$. We therefore place zero as the first figure in the decimal part of the root, and multiply the roots again by 10, before proceeding to the second transformation. 5 is easily seen to be a root of the equation thus transformed.

Ans. 7·05.

In the examples here considered the root terminates at an early stage. When the calculation is of greater length, if it were necessary to find the successive figures by substitution, the labour of the process would be very great. This, however, is not necessary, as will appear in the next Article; and one of the most valuable practical advantages of Horner's method is, that after the second, or third (sometimes even after the first) figure of the root is found, the *transformed equation itself suggests by mere inspection the next figure of the root.* The principle of this simplification will now be explained.

102. **Principle of the Trial-divisor.**—We have seen in Art. 100 that when an equation is transformed by the substitution of $a + h$ for x, a being a number differing from the true root by a quantity h small in proportion to a, an approximate numerical value of h is obtained by dividing $f(a)$ by $f'(a)$. Now the successive transformed equations in Horner's process are the results of transformations of this kind, the last coefficient being $f(a)$, and the second last $f'(a)$ (see Art. 33). Hence, after two or three steps have been completed, so that the part of the root remaining bears a small ratio to the part already evolved, we may expect to be furnished with two or three more figures of the root correctly by mere division of the last by the second last coefficient of the final transformed equation. We might therefore, if we pleased, at any stage of Horner's operations, apply Newton's method to get a further approximation to the root. In Horner's method this principle is employed to suggest the next following figure of the root after the figures already obtained. The second last coefficient of each transformed equation is called the *trial-divisor*. Thus, in the second example of the last Article, the number 5 is correctly suggested by the trial-divisor 2690800. In this example, indeed, the second figure of the root is correctly suggested by the trial-divisor of the first transformed equation; although, in general, such is not the case. In practice the student will have to estimate the probable effect of the leading coefficients of the transformed equation; he will find, however, that the influence of these terms becomes less and less as the evolution of the root proceeds.

Examples.

1. Find the positive root of the equation

$$x^3 + x^2 + x - 100 = 0$$

correct to four decimal places.

It is easily seen that the root lies between 4 and 5. We write down the work, and proceed to make observations on it:—

1	1	1	− 100	(4·2644
	4	20	84	
	5	21	**− 16000**	
	4	36	11928	
	9	**5700**	**− 4072000**	
	4	264	3788376	
	130	5964	**− 283624000**	
	2	268	256071744	
	132	**623200**	**− 27552256**	
	2	8196		
	134	631396		
	2	8232		
	1360	**63962800**		
	6	55136		
	1366	64017936		
	6	55152		
	1372	**64073088**		
	6			
	13780			
	4			
	13784			
	4			
	13788			
	4			
	13792			

First diminish the roots by 4. As the decimal part is now about to appear, attach ciphers to the coefficients of the transformed equation as explained in Ex. 2, Art. 101. Since the coefficient 130 is small in proportion to 5700, we may expect that the trial-divisor will give a good indication of the next figure. The figure to be adopted in every case as part of the root is *that highest number which in the process of transformation will not change the sign of the absolute term.* Here 2 is the proper figure. In diminishing by 2 the roots of the transformed equation

$$x^3 + 130x^2 + 5700x - 16000 = 0,$$

the absolute term retains its sign (− 4072). If we had adopted the figure 3, the absolute term would have become positive, the change of sign showing that we had gone beyond the root. We must take care that, after the first transformation (the reason of this restriction will appear in the next example), the absolute term preserves its sign throughout the operation. If we were to take by mistake a number too small, the error would show itself, just as in ordinary division or evolution by the next suggested number being greater than 9. Such a mistake, however, will rarely be made. The error which is most common is to take the number too large,

and this will show itself in the work by the change of sign in the absolute term. In the above work it is evident, without performing the fifth transformation, that the corresponding figure of the root is 4, so that the correct root to four decimal places is 4·2644.

2. The equation $$x^4 + 4x^3 - 4x^2 - 11x + 4 = 0$$

has one root between 1 and 2; find its value correct to four decimal places.

1	4	− 4	− 11	4	(1·6369
	1	5	1	− 10	
	5	1	10	− 60000	
	1	6	7	50976	
	6	7	− 3000	− 90240000	
	1	7	11496	72690561	
	7	1400	8496	− 175494390000	
	1	516	14808	152131052016	
	80	1916	23304000	− 23363337984	
	6	552	926187		
	86	2468	24230187		
	6	588	935601		
	92	305600	25165788000		
	6	3129	189387336		
	98	308729	25355175336		
	6	3138	189766488		
	1040	311867	25544941824		
	3	3147			
	1043	31501400			
	3	63156			
	1046	31564556			
	3	63192			
	1049	31627748			
	3	63228			
	10520	31690976			
	6				
	10526				
	6				
	10532				
	6				
	10538				
	6				
	10544				

We see without completing the fifth transformation that 9 is the next figure of the root. The root is, therefore, 1·6369 correct to four decimal places.

The trial-divisor becomes effective after the second transformation, suggesting correctly the number 3, and all subsequent numbers. The first transformed equation has its last two terms negative. We may expect, therefore, that the influence of the preceding coefficients is greater than that of the trial-divisor, as in fact is here the case. The number 6, the second figure of the root, must be found by substitution. We have to determine what is the situation between 0 and 10 of the root of the equation

$$x^4 + 80x^3 + 1400x^2 - 3000x - 60000 = 0.$$

A few trials show that 6 gives a negative, and 7 a positive result. Hence the root lies between 6 and 7; and 6 is the number of which we are in search. In the subsequent trials we take those greatest numbers 3, 6, 9, in succession, which allow the absolute term to retain its negative sign. In the first transformation, diminishing the roots by 1, there is a change of sign in the absolute term. The meaning of this is, that we have passed over a root lying between 0 and 1, for 0 gives a positive result, 4; and 1 gives a negative result, − 6. In all subsequent transformations, so long as we keep below the root, the sign of the absolute term must be the same as the sign resulting from the substitution of 1. This supposes of course that no root lies between 1 and that of which we are in search. This supposition we have already made in the statement of the question. In fact the proposed equation can have only two positive roots; one of them lies between 0 and 1, and therefore only one between 1 and 2.

When two roots exist between the limits employed in Horner's method, *i.e.* when the equation has a pair of roots nearly equal, certain precautions must be observed which will form the subject of a subsequent Article.

3. Find the root of the preceding equation between 0 and 1 to four decimal places. Commence by multiplying by 10. The coefficients are then

$$1, \; 40, \; -400, \; -11000, \; 40000;$$

the trial-divisor becomes effective at once in consequence of the comparative smallness of the leading coefficients. The positive sign of the absolute term must be preserved throughout. *Ans.* ·3373.

4. Find to three places of decimals the root situated between 9 and 10 of the equation

$$x^4 - 3x^2 + 75x - 10000 = 0.$$

[Supply the zero coefficient of x^3.] *Ans.* 9·886.

In the examples hitherto considered the root has been found to a few decimal places only. We proceed now to explain a method by which, after three or four places of decimals have been evolved as above, several more may be correctly obtained with great facility by a contracted process.

103. **Contraction of Horner's Process.**—In the ordinary process of contracted Division, when the given figures are exhausted, in place of appending ciphers to the successive dividends, we cut off figures successively from the right of the divisor, so that the divisor itself becomes exhausted after a number of steps depending on the number of figures it contains. The resulting quotient will differ from the true quotient in the last figure only, or at most in the last two figures. In Horner's contracted method the principle is the same. We retain those figures only which are effective in contributing to the result to the degree of approximation desired. When the contracted process commences, in place of appending ciphers to the successive coefficients of the transformed equation in the way before explained, we cut off one figure from the right of the last coefficient but one, two from the right of the last coefficient but two, three from the right of the last coefficient but three; and so on. The effect of this is to retain in their proper places the important figures in the work, and to banish altogether those which are of little importance.

The student will do well to compare the first transformation by the contracted process in the first of the following examples with the corresponding step in the second example of the last Article, where the transformation is exhibited in full. He will then observe how the leading figures (those which are most important in contributing to the result) coincide in both cases, and retain their relative places; while the figures of little importance are entirely dispensed with.

In addition to the contraction now explained, other abbreviations of Horner's process are sometimes recommended; but as the advantage to be derived from them is small, and as they increase the chances of error, we do not think it necessary to give any account of them. The contraction here explained is of so much importance in the practical application of Horner's method of approximation that no account of this method is complete without it.

Examples.

1. Find the root between 1 and 2 of the equation in Ex. 2 of the last Article correct to seven or eight decimal places.

Assuming the result of the Example referred to, we shall commence the contracted process after the third transformation has been completed. The subsequent work stands as follows :—

1~~052~~	3150~~14~~	2516578~~8~~	−17549439	(1·636913575
	6	18936	15213090	
	3156	2535515	−2336349	
	6	18972	2301597	
	3162	255448~~7~~	−34752	
	6	285	25601	
	31~~68~~	255733	−9151	
		285	7680	
	~~31~~	25601~~8~~	−1471	
			1280	
			−191	
			179	
			12	

Here the effect of the first cutting off of figures, namely, 8 from the second last coefficient, 14 from the third last, and 052 from the fourth last, is to banish altogether the first coefficient of the biquadratic. We proceed to diminish the roots by 6 as if the coefficients 1, 3150, 2516578, − 17549139 which are left were those of a cubic equation. In multiplying by the corresponding figure of the root the figures cut off should be multiplied mentally, and account taken of the number to be carried, just as in contracted division.

After the diminution by 6 has been completed, we cut off again in the transformed cubic 7 from the last coefficient but one, 68 from the last but two, and the first coefficient disappears altogether. The work then proceeds as if we were dealing with the coefficients 31, 255448, − 2336349 of a quadratic. The effect of the next process of cutting off is to banish altogether the leading coefficient 31. The subsequent work coincides with that of contracted division. When the operation terminates, the number of decimals in the quotient may be depended on up to the last two or three figures. The extent to which the evolution of the root must be carried before the contracted process is commenced depends on the number of decimal places required; for after the contraction commences we shall be furnished, in addition to the figures already evolved, with as many more as there are figures in the trial-divisor, less one.

2. Find to seven or eight decimal places the root of the equation

$$x^4 - 12x + 7 = 0$$

which lies between 2 and 3.

This equation can have only two positive roots : one lies between 0 and 1, and the other between 2 and 3. For the evolution of the latter we have the following :—

1	0	0	− 12	7	(2·0472755671
	2	4	8	−	
	2	4	− 4	− 100000000	
	2	8	24	83891456	
	4	12	20000000	− 16108544	
	2	12	972864	15493401	
	6	240000	20972864	− 615143	
	2	3216	985792	446262	
	800	243216	2195865~~6~~	− 168881	
	4	3232	17478	156226	
	804	246448	2213343	− 12655	
	4	3248	17478	11159	
	808	2496~~96~~	223082~~1~~	− 1496	
	4	24~~96~~	49	1338	
	812		223131	− 158	
	4		49	156	
	8~~16~~	~~24~~	22318~~0~~	2	

On this we remark, that after diminishing the roots by 2, and multiplying the roots of the transformed equation by 10, we find that the trial-divisor 20000 will not "go into" the absolute term 10000 ; we put, therefore, zero in the quotient, and multiply again by 10, and then proceed as before.

3. Find the root of the same equation which lies between 0 and 1.

Ans. ·593685829.

4. Find the positive root of the equation

$$x^3 + 24{\cdot}84x^2 - 67{\cdot}613x - 3761{\cdot}2758 = 0.$$

[When the coefficients of the proposed equation contain decimal points, it will be found that they soon disappear in the work in consequence of the multiplications by 10 after the decimal part of the root begins to appear.

Ans. 11·1973222.

5. Find the negative root of the equation

$$x^4 - 12x^2 + 12x - 3 = 0$$

to seven places of decimals.

When a negative root has to be found, it is convenient to change the sign of x and find the corresponding positive root of the transformed equation.

Ans. − 3·9073785.

104. **Application of Horner's Method to Cases where Roots are nearly Equal.**—We have seen in Art. 100 that the method of approximation there explained fails when the proposed equation has two roots nearly equal. Examples of this nature are those which present most difficulties, both in their analysis (see Ex. 7, Art. 91) and in their solution. By Horner's method it is possible, with very little more labour than is necessary in other cases, to effect the solution of such equations. So long as the leading figures of the two roots are the same certain precautions must be observed, which will be illustrated by the following examples. After the two roots have been separated, the subsequent calculation proceeds for each root separately, just as in the examples of the previous Articles. It is evident, from the explanation of the trial-divisor given in Art. 102, that for the same reason as that which explains the failure of Newton's method in the case under consideration (see Art. 100), it will not become effective till the first or second stage after the roots have been separated.

EXAMPLES.

1. The equation

$$x^3 - 7x + 7 = 0$$

has two roots between 1 and 2 (see Ex. 2, Art. 89); find each of them to eight decimal places.

Diminishing the roots by 1, we find that the transformed equation (after its roots are multiplied by 10), viz.

$$x^3 + 30x^2 - 400x + 1000 = 0,$$

must have two roots between 0 and 10. We find that these roots lie, one between 3 and 4, and the other between 6 and 7. The roots are now separated, and we proceed with each separately in the manner already explained. If the roots were not separated at this stage, we should find the leading figure common to the two, and, having diminished the roots by it, find in what intervals the roots of the resulting equation were situated; and so on.

Ans. 1·35689584, 1·69202147.

2. Find the two roots of the equation

$$x^3 - 49x^2 + 658x - 1379 = 0$$

which lie between 20 and 30.

We shall exhibit the complete work of approximation to the smaller of the two roots to seven places; and then make certain observations which will be a guide to the student in all cases of the kind.

1	− 49	658	− 1379	(23·2131277
	20	− 580	1560	
	− 29	78	**181**	
	20	− 180	− 180	
	− 9	**− 102**	**1000**	
	20	42	− 992	
	11	− 60	**8000**	
	3	51	− 6739	
	14	**− 900**	**1261000**	
	3	404	− 1217403	
	17	− 496	43597	
	3	408	− 34183	
	200	**− 8800**	9414	
	2	2061	− 6786	
	202	− 6739	2628	
	2	2062	− 2372	
	204	**− 467700**	256	
	2	61899	− 236	
	2060	− 405801	20	
	1	61908		
	2061	− 34389$\cancel{3}$		
	1	206		
	2062	− 34183		
	1	206		
	20630	2$\cancel{0}\cancel{6}$ − 3397$\cancel{7}$		
	3	4		
	20633	− 3393		
	3	4		
	20636	$\cancel{2}$ − 338$\cancel{9}$		
	3			
	206$\cancel{3}\cancel{9}$			

The diminution of the roots by 20 changes the sign of the absolute term. This is an indication that a root exists between 0 and 20, with which we are not at present concerned. The roots of the first transformed equation

$$x^3 + 11x^2 - 102x + 181 = 0$$

are not yet separated, lying both between 3 and 4. The substitution of each of these numbers gives a positive result, so that we have not here the same criterion to guide us in our search for the proper figure as in former cases, viz. a change of sign in the absolute term. We have, however, a different criterion which enables

us to find by mere substitution the interval within which the two roots lie. If we diminish the roots of $x^3 + 11x^2 - 102x + 181 = 0$ by 4, the resulting equation is $x^3 + 23x^2 + 34x + 13 = 0$, which has no change of sign. Hence the two roots must lie between 0 and 4. If we diminish its roots by 3, the resulting equation (as in the above work) has the same number of changes of sign as the equation itself. Hence the two roots lie between 3 and 4. They are, therefore, not yet separated; and we proceed to diminish by 3. The next transformed equation

$$x^3 + 200x^2 - 900x + 1000 = 0$$

is found in the same way to have both its roots between 2 and 3; the diminution by 2 leaving two changes of sign in the coefficients of the transformed equation (as in the above work), and the diminution by 3 giving all positive signs. So far, then, the two roots agree in their first three figures, viz. 23·2. We diminish again by 2. The resulting equation $x^3 + 2060x^2 - 8800x + 1261000 = 0$ has one root only between 1 and 2; 1 giving a positive, and 2 a negative result: its other root lies between 2 and 3; 3 giving a positive result. The roots are now separated. We proceed, as in the above work, to approximate to the lesser root, by diminishing the roots of this equation by 1; the trial divisor becoming effective at the next step. To approximate to the greater root, we must diminish by 2 the roots of the same equation, taking care that in the subsequent operations the negative sign, to which the previously positive sign of the absolute term now changes, is preserved. The second root will be found to be 23·2295212.

So long as the two roots remain together, a guide to the proper figure of the root may be obtained by dividing twice the last coefficient by the second last, or the second last by twice the third last. The reason of this is, that the proposed equation approximates now to the quadratic formed by the last three terms in each transformed equation, just as in previous cases, and in Newton's method, it approximated to the simple equation formed by the last two terms, this quadratic having the two nearly equal roots for its roots; and when the two roots of the equation $ax^2 + bx + c = 0$ are nearly equal, either of them is given approximately by $\dfrac{-2c}{b}$ or $\dfrac{-b}{2a}$. Thus, in the above example, the number 3 is suggested by $\dfrac{2 \times 181}{102}$, and the number 2 by $\dfrac{2 \times 1000}{900}$. In this way we can generally, at the first attempt, find the two integers between which the pair of roots lies. We shall have also an indication of the separation of the roots by observing when the numbers suggested in this way by the last three coefficients become different, *i. e.* when $\dfrac{2c}{b}$ suggests a different number from $\dfrac{b}{2a}$.

3. Calculate to three decimal places each of the roots lying between 4 and 5 of the equation

$$x^4 + 8x^3 - 70x^2 - 144x + 936 = 0.$$

Ans. 4·242; 4·246.

4. Find the two roots between 2 and 3 of the equation

$$64x^3 - 592x^2 + 1649x - 1445 = 0.$$

Ans. The roots are both = 2·125.

Here we find that the two roots are not separated at the third decimal place. When we diminish by 5 the absolute term vanishes, showing that 2·125 is a root; and proceeding with this diminution the second last coefficient also vanishes. Hence 2·125 is a double root.

When an equation contains more than two nearly equal roots, they can all be found by Horner's process in a manner similar to that now explained. Such cases are, however, of rare occurrence in practice. The principles already laid down will be a sufficient guide to the student in all cases of the kind.

105. **Lagrange's Method of Approximation.**—Lagrange has given a method of expressing the root of a numerical equation in the form of a continued fraction. As this method is, for practical purposes, much inferior to that of Horner, we shall content ourselves with a brief account of it.

Let the equation $f(x) = 0$ have one root, and only one root, between the two consecutive integers a and $a + 1$. Substitute $a + \frac{1}{y}$ for x in the proposed equation. The transformed equation in y has one positive root. Let this be determined by trial to lie between the integers b and $b + 1$. Transform the equation in y by the substitution $y = b + \frac{1}{z}$. The positive root of the equation in z is found by trial to lie between c and $c + 1$. Continuing this process, an approximation to the root is obtained in the form of a continued fraction, as follows:—

$$a + \cfrac{1}{b + \cfrac{1}{c + 1 \ldots}}$$

EXAMPLES.

1. Find in the form of a continued fraction the positive root of the equation

$$x^3 - 2x - 5 = 0.$$

The root lies between 2 and 3.

To make the transformation $x = 2 + \frac{1}{y}$, we first employ the process of Art. 33, diminishing the roots by 2. We then find the equation whose roots are the reciprocals of the roots of the transformed.

The equation in y is in this way found to be

$$y^3 - 10y^2 - 6y - 1 = 0.$$

This has a root between 10 and 11.

Make now the substitution $y = 10 + \dfrac{1}{z}\cdot$

The equation in z is

$$61z^3 - 94z^2 - 20z - 1 = 0.$$

This has a root between 1 and 2. Take $z = 1 + \dfrac{1}{u}\cdot$

The equation in u is

$$54u^3 + 25u^2 - 89u - 61 = 0,$$

which has a root between 1 and 2; and so on.

We have, therefore, the following expression for the root

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

2. Find in the form of a continued fraction the positive root of

$$x^3 - 6x - 13 = 0.$$

$$\textit{Ans.}\ 3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

106. **Numerical Solution of the Biquadratic.**—It is proper, before closing the subject of the solution of numerical equations, to illustrate the practical uses which may be made of the methods of solution of Chap. VI. Although, as before observed, the numerical solution of equations is in general best effected by the methods of the present chapter, there are certain cases in which it is convenient to employ the methods of Chap. VI. for the resolution of the biquadratic. When a biquadratic equation leads to a reducing cubic which has a commensurable root, this root can be readily found, and the solution of the biquadratic completed. We proceed to solve a few examples of this kind, using Descartes' method (Art. 64), which will usually be found the most convenient in practice.

Examples.

1. Resolve the quartic

$$x^4 - 6x^3 + 3x^2 + 22x - 6$$

into quadratic factors.

Making the assumption of Art. 64, we easily obtain

$$+p' = -3, \quad q + q' + 4pp' = 3, \quad pq' + p'q = 11, \quad qq' = -6.$$

Also $$\phi = \frac{1}{2} - pp' = \frac{1}{4}(q + q' - 1),$$

and, calculating I and J, the equation for ϕ is

$$4\phi^3 - \frac{111}{4}\phi - \frac{225}{8} = 0.$$

Multiplying the roots by 4, we have, if $4\phi = t$,

$$t^3 - 111t - 450 = 0.$$

By the Method of Divisors this is easily found to have a root -6; hence $\phi = -\frac{3}{2}$, giving $$pp' = 2, \quad q + q' = -5.$$

From these, combined with the preceding equations, we get

$$p = -2, \quad p' = -1, \quad q = 1, \quad q' = -6.$$

When the values of q und q' are found, the equation giving the value of $pq' + p'q$ determines which value of q goes with p, and which with p', in the quadratic factors. The quartic is resolved, therefore, into the factors

$$(x^2 - 4x + 1)(x^2 - 2x - 6).$$

By means of the other two values of ϕ we can resolve the quartic into quadratic factors in two other ways; or we can do the same thing by solving the two quadratics already obtained.

2. Resolve into factors the quartic

$$f(x) \equiv x^4 - 8x^3 - 12x^2 + 60x + 63.$$

The equation for ϕ is

$$4\phi^3 - 195\phi - 475 = 0,$$

which is found to have a root $= -5$.

Ans. $f(x) \equiv (x^2 - 2x - 3)(x^2 - 6x - 21)$.

3. Resolve into factors

$$f(x) \equiv x^4 - 17x^2 - 20x - 6.$$

The reducing cubic is found to be

$$4\phi^3 - \frac{217}{12}\phi + \frac{3185}{216} = 0;$$

or, multiplying the roots by 6,

$$4t^3 - 651t + 3185 = 0.$$

This has a root $= 7$; hence $\phi = \frac{7}{6}$.

Ans. $f(x) \equiv (x^2 + 4x + 2)(x^2 - 4x - 3)$.

4. Resolve into factors

$$f(x) \equiv x^4 - 6x^3 - 9x^2 + 66x - 22.$$

The reducing cubic is

$$4\phi^3 - \frac{335}{4}\phi - \frac{897}{8} = 0;$$

hence
$$\phi = -\frac{3}{2}.$$

Ans. $f(x) \equiv (x^2 - 11)(x^2 - 6x + 2)$.

5. Resolve into factors

$$f(x) \equiv x^4 - 8x^3 + 21x^2 - 26x + 14.$$

Ans. $f(x) \equiv (x^2 - 2x + 2)(x^2 - 6x + 7)$.

6. Resolve into factors

$$x^4 + 12x + 3.$$

Ans. $(x^2 - x\sqrt{6} + 3 + \sqrt{6})(x^2 + x\sqrt{6} + 3 - \sqrt{6})$.

7. Find the quadratic factors of

$$x^4 - 8x^3 - 12x^2 + 84x - 63 = 0,$$

and solve the equation completely (see Ex. 18, p. 34).

Ans. $\{x^2 - 2x(2 + \sqrt{7}) + 3\sqrt{7}\}\{x^2 - 2x(2 - \sqrt{7}) - 3\sqrt{7}\}$.

Miscellaneous Examples.

1. Find the positive root of

$$x^3 - 6x - 13 = 0.$$

Ans. 3·176814393.

2. Find the positive root of

$$x^3 - 2x - 5 = 0$$

correct to eight or nine places. *Ans.* 2·094551483.

3. The equation

$$2x^3 - 650{\cdot}8x^2 + 5x - 1627 = 0$$

has a root between 300 and 400: find it.

Ans. Commensurable root, 325·4.

4. Find the root between 20 and 30 of the equation

$$4x^3 - 180x^2 + 1896x - 457 = 0.$$

Ans. 28·52127738.

5. Find to six places the root between 2 and 3 of the equation

$$x^3 - 49x^2 + 658x - 1379 = 0.$$

Ans. 2·557351.

6. Find to six places the root between 2 and 3 of the equation

$$x^4 - 12x^2 + 12x - 3 = 0.$$

Ans. 2·858083.

7. Find the positive root of the equation

$$x^3 + 2x^2 - 23x - 70 = 0$$

correct to about ten decimal places. *Ans.* 5·13457872528.

8. Find the cube root of 673373097125. *Ans.* 8765.

9. Find the fifth root of 537824. *Ans.* 14.

10. Find all the roots of the cubic equation

$$x^3 - 3x + 1 = 0.$$

The equation $x^6 + x^3 + 1 = 0$, of Ex. 7, p. 100, reduces to this.

Ans. $-1{\cdot}87938$, $\cdot34729$, $1{\cdot}53209$.

The smaller positive root gives the solution of the problem—To divide a hemisphere whose radius is unity into two equal parts by a plane parallel to the base.

11. Find all the roots of the cubic

$$x^3 + x^2 - 2x - 1 = 0.$$

(See Ex. 1, p. 100.) *Ans.* $-1{\cdot}80194$, $-{\cdot}44504$, $1{\cdot}24698$.

12. Find to five decimal places the negative root between -1 and 0 (see Ex. 3, p. 100) of the equation

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0.$$

Ans. $-\cdot 28463$.

13. Solve the equation

$$x^3 - 315x^2 - 19684x + 2977260 = 0.$$

We here find that a root exists between 70 and 80. By Horner's process it is found to be 78. The depressed equation furnishes two roots, which, increased by 78, are the other roots of the cubic.

Ans. 78, 347, -110.

14. Find the two real roots of the equation

$$x^4 - 11727x + 40385 = 0.$$

Ans. 3·45592, 21·43067.

This equation is given by Mr. G. H. Darwin in a Paper *On the Precession of a Viscous Spheroid, and on the Remote History of the Earth. Phil. Trans.*, Part ii., 1879, p. 508. The roots are "the two values of the cube root of the earth's rotation for which the earth and moon move round as a rigid body."

15. Find all the roots of the cubic equation

$$20x^3 - 24x^2 + 3 = 0.$$

Ans. $-0\cdot 31469$, 0·44603, 1·06865.

This equation occurs in the solution by Professor Ball of a problem of Professor Townsend's in the *Educational Times* of Dec., 1878, to determine the deflection of a beam uniformly loaded and supported at its two ends and points of trisection.

16. Find the positive root of the equation

$$14x^3 + 12x^2 - 9x - 10 = 0.$$

Ans. 0·85906.

The equations of this and the following example occur in the investigation of questions relative to beams supported by props.

17. Find the positive root of the equation

$$7x^4 + 20x^3 + 3x^2 - 16x - 8 = 0.$$

Ans. 0·91336.

18. Find to ten decimal places the positive root of the equation

$$x^5 + 12x^4 + 59x^3 + 150x^2 + 201x - 207 = 0.$$

Ans. ·6386058033.

19. Find all the commensurable roots of

$$f(x) \equiv x^5 + 2x^4 - 36x^3 - 149x^2 - 232x - 336 = 0,$$

and solve the equation completely.

Ans. $f(x) \equiv (x^2 + x + 3)(x + 4)^2 (x - 7)$.

20. Solve similarly the equation

$$f(x) \equiv x^5 - 32x^4 + 116x^3 - 116x^2 + 115x - 84 = 0.$$

Ans. $f(x) \equiv (x^2 + 1)(x - 1)(x - 3)(x - 28)$.

21. Apply the method of Ex. 21, p. 199, to the analysis of the equation of Ex. 1, Art. 91.

Disregarding the two lowest Sturmian remainders, we have

$$f(x) \equiv x^4 + 3x^3 + 7x^2 + 10x + 1,$$
$$f'(x) \equiv 4x^3 + 9x^2 + 14x + 10,$$
$$R_1 \equiv - 29x^2 - 78x + 14.$$

The roots of the equation $R_1 = 0$ are easily seen to lie in the intervals $(-3, -2)$ and $(0, 1)$. The equation $f(x) = 0$ has two imaginary roots, since the coefficient of x^2 in R_1 is negative. The real roots, if any, must be negative. The three functions above written are sufficient to determine the existence and situations of roots in the intervals $(-\infty, -3)$ and $(-2, 0)$. It is at once seen that two real roots of the original equation are situated in the latter interval.

It will be found possible in many examples to avoid in this way the calculation of the last two Sturmian remainders; and it will be observed that it is not necessary to know the actual roots of the quadratic function, but only the intervals in which they are situated.

22. In the application of Sturm's theorem, if any function be reached whose signs are all positive or all negative, the number and situations of the positive roots of the original equation can be examined without the aid of the lower Sturmians; and if a function be reached whose signs are alternately positive and negative, the negative roots of the original equation may be discussed in a similar manner.

23. Find the condition that the quadratic Sturmian remainder of Ex. 3, Art. 92, should have its roots imaginary.

Ans. $HI + 3aJ$ positive.

This condition is fulfilled when H and J are both positive (since then I must be positive, by the identity of Art. 37). It is, therefore, easily inferred that the biquadratic has no real roots when H and J are both positive (cf. Ex. 11, p. 198).

24. When the biquadratic has two roots equal to α, prove

$$a\alpha + b = \frac{-GI}{2HI - 3aJ}.$$

25. If the equation $f(x) = 0$ has all its roots real, prove that the equation $f(x) f''(x) - [f'(x)]^2 = 0$ has all its roots imaginary.

26. Calculate the leading coefficients of the first two Sturmian remainders for an equation of the n^{th} degree wanting the second term, viz.

$$x^n + ax^{n-2} + bx^{n-3} + cx^{n-4} + \&c. = 0.$$

No coefficients beyond those here given will enter into the required values: we readily find

$$R_1 \equiv -2ax^{n-2} - 3bx^{n-3} - 4cx^{n-4} - \&c.$$

$$R_2 \equiv -\{4(n-2)a^3 - 8nac + 9nb^2\}x^{n-3} + \&c.$$

(cf. Ex. 12, p. 198.)

27. Remove the second term from the general equation of the n^{th} degree written with binomial coefficients, and prove that the leading coefficients of the first two Sturmian remainders of the resulting equation are

$$-H, \quad -nHI + 3(n-2)a_0J.$$

These expressions are easily derived from the preceding example by aid of the transformation of Art. 35; the values of A_2, A_3, A_4 being given by the equations

$$a_0A_2 = H, \quad a_0^2A_3 = G, \quad a_0^3A_4 = a_0^2I - 3H^2,$$

G^2 being replaced by its value from the identity of Art. 37, and positive multipliers omitted (cf. Ex. 13, p. 198).

28. Calculate Sturm's functions for Euler's cubic (see Art. 61).

We find, after some reductions, and omitting positive factors,

$$f(x) \equiv x^3 + 3Hx^2 + 3(H^2 - \tfrac{1}{12}a^2I)x - \tfrac{1}{4}G^2,$$

$$f'(x) \equiv x^2 + 2Hx + H^2 - \tfrac{1}{12}a^2I,$$

$$R_1 \equiv 2Ix + 2HI - 3aJ,$$

$$R_2 \equiv I^3 - 27J^2.$$

All the conditions of Art. 68, with respect to the nature of the roots of the biquadratic, may be derived from these results, by the aid of Ex. 4, p. 125. And it will be observed that the conditions for reality of all the roots as given in Art. 93, as well as in the article already referred to, are both obtained here together; for, in order that Euler's cubic should have all its roots real and positive, the substitution of 0 for x must give three changes of sign, and this requires that $a^2I - 12H^2$ and $2HI - 3aJ$ should be both negative.

CHAPTER XI.

COMPLEX NUMBERS AND THE COMPLEX VARIABLE.

107. **Complex Numbers—Graphic Representation.** —In the foregoing chapters many examples have been met with of the occurrence among the solutions of numerical equations of quantities of the form $a + b\sqrt{-1}$, involving the extraction of the square root of a negative number. Such an expression, consisting of a positive or negative real units and b positive or negative imaginary units, is called a *complex number* (see Art. 15). The imaginary unit $\sqrt{-1}$ is denoted for brevity by i. Real and purely imaginary numbers are both included in the expression $a + ib$, the former being obtained when $b = 0$, and the latter when $a = 0$. Complex numbers may be submitted to all the ordinary rules of arithmetical calculation; and in the result of any such calculation integral powers of i beyond the first can always be reduced by the relation $i^2 = -1$.

We proceed to explain a mode of representing complex numbers geometrically, which will be found very convenient in the treatment of functions involving quantities of this kind.

The expression $a + ib$ may be written in the form

$$\mu(\cos\alpha + i\sin\alpha),$$

where

$$\mu = \sqrt{a^2 + b^2}, \quad \cos\alpha = \frac{a}{\mu}, \quad \sin\alpha = \frac{b}{\mu}.$$

The quantity μ is called the *modulus*, and the angle α the *argument* of the complex number $a + ib$. The modulus is always taken positively, the negative sign of the radical corresponding to an increase of the argument by π.

Let rectangular axes OX, OY (fig. 7) be taken; and a point A such that $XOA = \alpha$, and $OA = \mu$. We have then $OM = \mu \cos \alpha = a$, and $AM = \mu \sin \alpha = b$. The expression $a + ib$ may therefore be represented graphically by the right line drawn from O to a point in a plane whose co-ordinates referred to the fixed axes are a, b; the distance OA of this point from the origin being equal to the modulus, and the angle XOA equal to the argument of the complex number.

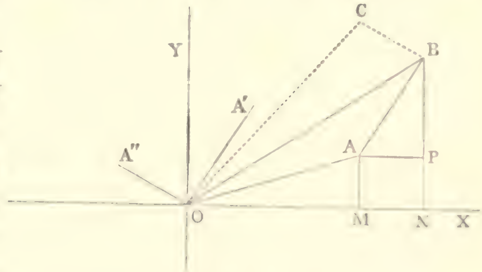

Fig. 7.

The magnitude of a complex quantity is estimated by the magnitude of its modulus. When the complex quantity vanishes (that is, when a and b separately vanish) its modulus vanishes; and, conversely, when the modulus vanishes, since then $a^2 + b^2 = 0$, a and b must separately vanish, and therefore the complex quantity itself. Two such quantities, $a + ib$ and $a' + ib'$, are equal when $a = a'$ and $b = b'$, i.e. when the moduli are equal and when the arguments either are equal or differ by a multiple of 2π.

In what follows we shall for brevity represent the modulus and argument of $a + ib$ by the notation

$$mod.\ (a + ib), \quad arg.\ (a + ib).$$

108. **Complex Numbers.—Addition and Subtraction.**—Let a second complex number $a' + ib'$ be represented by the right line OA', so that

$$OA' = mod.\ (a' + ib'), \quad XOA' = arg.\ (a' + ib').$$

We proceed to determine the mode of representing the sum

$$a + ib + a' + ib'.$$

Writing this sum in the form $a + a' + i(b + b')$, we observe, in accordance with the convention of Art. 107, that it will be represented by the line drawn from the origin to the point whose co-ordinates are $a + a'$, $b + b'$. To find this point, draw AB parallel and equal to OA'; since AP, BP, are equal to a', b', B is the required point, and we have

$$OB = mod. \{a + a' + i(b + b')\}, \quad XOB = arg. \{a + a' + i(b + b')\}.$$

To add two complex numbers, therefore, we draw OA to represent one of them; and, at its extremity, AB to represent the second (that is, so that its length is equal to the modulus, and the angle it makes with OX equal to the argument, of the second); then OB represents the sum of the two complex numbers.

Since OB is not greater than $OA + AB$, it follows that *the modulus of the sum of two complex numbers is less than* (*or at most equal to*) *the sum of their moduli.*

This mode of representation may be extended to the addition of any number of such quantities. Thus, to add a third $a'' + ib''$, represented by OA'', we draw BC parallel and equal to OA'', and join OC. Then OC represents the sum of the three, OA, OA', OA''. It is evident also that we may conclude in general that *the modulus of the sum of any number of complex quantities is less than* (*or at most equal to*) *the sum of their moduli.*

Subtraction can be represented in a similar way. Since OB represents the sum of OA and OA', OA will represent the difference of OB and OA'. To subtract two complex numbers, therefore, we draw at the extremity of the line representing the first a line parallel and equal to the second, but in an opposite direction (*i.e.* a direction which makes with OX an angle greater by π than the argument of the second). We join O to the extremity of this line to find the right line which represents the difference of the two given complex numbers.

109. **Multiplication and Division.**—To multiply the two complex numbers $a + ib$, $a' + ib'$, we write them in the form

$$a + ib = \mu(\cos \alpha + i \sin \alpha), \quad a' + ib' = \mu'(\cos \alpha' + i \sin \alpha').$$

We have then, by De Moivre's theorem,

$$(a+ib)(a'+ib') = \mu\mu' \{\cos(\alpha+\alpha') + i\sin(\alpha+\alpha')\},$$

which proves that *the product of two complex numbers is a complex number, whose modulus is the product of the two moduli, and whose argument is the sum of the two arguments.*

In the same way it appears that the product of any number of such factors is a complex quantity, whose modulus is the product of all the moduli, and whose argument is the sum of all the arguments.

To divide $a + ib$ by $a' + ib'$, we have similarly

$$\frac{a+ib}{a'+ib'} = \frac{\mu}{\mu'}\{\cos(\alpha-\alpha') + i\sin(\alpha-\alpha')\},$$

which proves that *the quotient of two complex numbers is a complex number, whose modulus is the quotient of the two moduli, and whose argument is the difference of the two arguments.*

It was assumed in the proof of the theorem of Art. 16 that when a product of any number of factors (real or imaginary) vanishes, one of the factors must vanish. This is evident when the factors are all real. From what is above proved the same conclusion holds when the factors are complex; for, in order that the modulus of the product may vanish, one of its factors must vanish, and therefore the complex quantity of which that factor is the modulus.

110. **Other Operations on Complex Numbers.**—From the foregoing propositions it follows that any integral power of a complex number, *e.g.* $(a + ib)^m$, can be expressed in the form $A + iB$, where A and B are real. And, more generally, if in any rational integral function

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n,$$

whose coefficients are complex (including real) numbers, a

complex quantity $a + ib$ be substituted for z, the result can be expressed in the standard form $A + iB$.

It is not proposed in the present chapter to discuss any functions of complex numbers beyond the rational integral function of the kind hitherto treated in this work. It is easy, however, to show, by the aid of De Moivre's theorem, that the remaining processes of numerical calculation—powers with fractional or complex exponents, logarithms, and powers whose base and exponent are both complex—reproduce in every case a complex number as result. This is expressed by saying that complex numbers form a system or group complete in themselves.

111. **The Complex Variable.**—In the earlier chapters of the present work the variation of a polynomial was studied corresponding to the passage of the variable through real values from $-\infty$ to $+\infty$; and the mode of representing by a figure the form of the polynomial was explained. Such a mode of treatment is only a particular case of a more general inquiry. Given a polynomial, rational and integral in z, whose coefficients are numbers real or complex, viz.

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n,$$

we may study its variations corresponding to the different values of z, where z has the complex form $x + iy$, and where x and y both take all possible real values. This form $x + iy$ is called the *complex variable.* All possible *real* values of the variable are of course included in the values of $x + iy$, being those values which arise by varying x and putting $y = 0$. In accordance with the principles of Art. 107 we may represent the complex variable $x + iy$ by the line OP (fig. 8) drawn from a fixed origin O to the point whose coordinates are x, y. Or we may say, $x + iy$ is represented by the point P. Thus all possible values of $x + iy$ will be represented by all the points in a plane. Since for any particular value of z, $f(z)$ takes the form $A + iB$ (Art. 110), the values of $f(z)$ may be represented in a similar manner by points

in a plane. We confine ourselves in the present Article to the representation of the variable $x + iy$ itself. We conceive the variation of $x + iy$ to take place in a continuous manner; for example, by the motion of the point x, y, along a curve. If OP and OP' represent two consecutive values of the variable, we write the corresponding values $x + iy$, $x' + iy'$, as follows:—

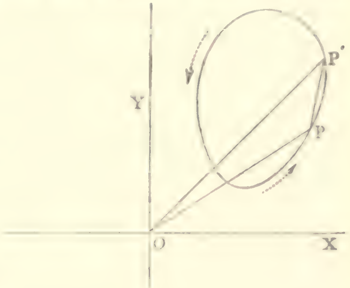

Fig. 8.

$$z = x + iy = r(\cos\theta + i\sin\theta), \quad z' = x' + iy' = r'(\cos\theta' + i\sin\theta').$$

Since OP' represents the sum of OP and PP' (Art. 108), it follows that PP' represents the increment of z; and if $z' = z + h$, h may be written in the form

$$h = \rho(\cos\phi + i\sin\phi),$$

where $\rho = PP'$, and ϕ is the angle PP' makes with OX.

The variation of the modulus of z is $OP' - OP$ or $r' - r$; the variation of the argument of z is $P'OP$ or $\theta' - \theta$; the variation of z itself is h or $\rho(\cos\phi + i\sin\phi)$, as just explained.

Let the point be supposed to describe a closed curve. When it returns to its original position P, the modulus takes again its original value; and the argument takes its original value if the point O is exterior to the curve, or is increased by 2π if O is interior to the curve.

If the complex variable describes the same line in two opposite directions, the variations of its argument are equal and of opposite signs, *i.e.* the total variation is nothing. From this we can derive a property of the variation of the argument of the complex variable, which will be found of importance in our succeeding investigations.

Let a plane area be divided into any number of parts by lines BD, AF, EC, &c. (fig. 9); then *the variation of the argument*

relatively to the perimeter of the whole area is equal to the sum of its variations relatively to the perimeters of the partial areas: all the areas being supposed to be described by the variable moving in the same sense. This is evident; for when the point is made to describe all the partial areas in the same sense,

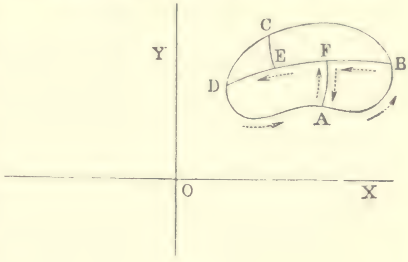

Fig. 9.

each of the internal dividing lines will be described twice, the two descriptions being in opposite directions; and the external perimeter will be described once; hence the total variation of the argument relatively to the dividing lines vanishes, and the variation relatively to the external perimeter alone remains. Take, for example, the areas ABF, AFD in the figure. When the point describes these areas in the sense indicated by the arrows, the total variation relatively to the line AF vanishes.

112. **Continuity of a Function of the Complex Variable.**—Suppose the complex variable z, starting from a fixed value z_0, to receive a small increment $h = \rho\,(\cos\phi + i\sin\phi)$; we have then, if $f(z)$ be the given function, replacing x by z in the expansion of Art. 6,

$$f(z) = f(z_0 + h) = f(z_0) + f'(z_0)\,h + \frac{f''(z_0)}{1\,.\,2}\,h^2 + \&c.,$$

and the increment of $f(z)$, being equal to $f(z_0 + h) - f(z_0)$, is

$$f'(z_0)\,h + \frac{f''(z_0)}{1\,.\,2}\,h^2 + \frac{f'''(z_0)}{1\,.\,2\,.\,3}\,h^3 + \&c. \ldots$$

In this expression the coefficients of the powers of h are all complex expressions of the usual form; and if their moduli be a, b, c, &c., the moduli of the successive terms are $a\rho$, $b\rho^2$, $c\rho^3$, &c.; and since, by Art. 108, the modulus of a sum is less than the sum of the moduli, it follows that the modulus of the increment of $f(z)$ is less than

$$a\rho + b\rho^2 + c\rho^3 + \&c.$$

Now a value may be assigned to ρ (Art. 5), such that for it or any smaller value, the value of this expression will be less than any assigned quantity. It follows that to an infinitely small variation of the complex variable (viz. one whose modulus is infinitely small) corresponds an infinitely small variation of the function; in other words, *the function varies continuously at the same time as the complex variable itself.*

113. **Variation of the Argument of $f(z)$ corresponding to the description of a small Closed Curve by the Complex Variable.**—Corresponding to a continuous series of values of z we have a continuous series of values of $f(z)$, which can be represented, like the values of z itself, by points in a plane. We represent these series of points by two figures (fig. 10) side

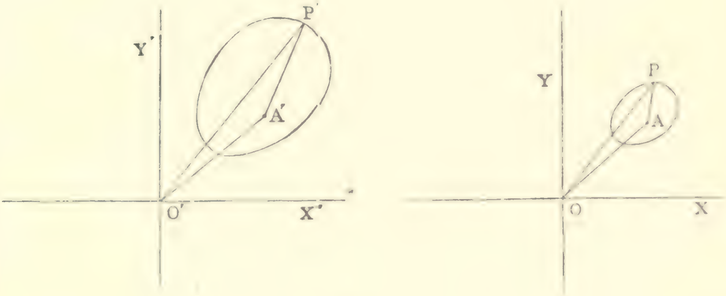

Fig. 10.

by side, which, to avoid confusion, may be supposed to be drawn on different planes. To each point P, representing $x + iy$, corresponds one determinate point P' representing $f(z)$. When P describes a continuous curve, P' describes also a continuous curve; and when P returns to its original position after describing a closed curve, P' returns also to its original position.

Our present object is to discuss the variation of the argument of $f(z)$ corresponding to the description of a small closed curve by P. Let A be any determinate point whose co-ordinates are

x_0, y_0, i.e. $z_0 = x_0 + iy_0$. We divide the discussion into two cases:—

(1). When $x_0 + iy_0$ is not a root of $f(z) = 0$, *i.e.* when $f(z_0)$ is different from zero.

(2). When $x_0 + iy_0$ is a root of $f(z) = 0$, or $f(z_0) = 0$.

(1). In the first case, to the point A corresponds a point A' representing the value of $f(z_0)$, and $O'A'$ is different from zero. Let $z = z_0 + h$, where $h = \rho(\cos\phi + i\sin\phi)$; and suppose P, which represents z, to describe a small closed curve round A. Let P' represent $f(z)$; then $A'P'$ represents the increment of $f(z)$ corresponding to the increment AP of z. By the previous Article it appears that values so small may be assigned to ρ, that the modulus of the increment of $f(z)$, namely $A'P'$, may be always less than the assigned quantity $O'A'$; hence P may be supposed to describe round A a closed curve so small that the corresponding closed curve described by P' will be exterior to O'. It follows, by Art. 111, that *corresponding to the description by* P *of a small closed curve, which does not contain a point satisfying the equation* $f(z) = 0$, *the total variation of the argument of* $f(z)$ *is nothing.*

(2). In the second case, suppose $x_0 + iy_0$ is a root of the equation $f(z) = 0$ repeated m times, and let

$$f(z) \equiv (z - z_0)^m \psi(z);$$

then

$$f(z) = h^m \psi(z) = \rho^m (\cos m\phi + i \sin m\phi)\, \psi(z).$$

In this case $O'A' = 0$; and when P describes a closed curve round A, P' returns to its original position, and the argument of $f(z)$ will be increased by a multiple of 2π, which may be determined as follows:—From the above equation we have

$$arg.\, f(z) = m\phi + arg.\, \psi(z);$$

and the increment of $arg.\, f(z)$ will be obtained by adding the increment of $m\phi$ to the increment of $arg.\, \psi(z)$. Now the latter

increment is nothing by (1), since the curve described by P may be supposed to contain no root of $\psi(z) = 0$; and since the increment of ϕ is 2π in one revolution of P, the increment of $m\phi$ is $2m\pi$. It follows that *when P describes a small closed curve containing a root of the equation $f(z) = 0$, repeated m times, the argument of $f(z)$ is increased by $2m\pi$.*

114. **Cauchy's Theorem.**—When z describes the same line in a plane in two opposite directions, $f(z)$ describes the corresponding line in its plane in two opposite directions, and the *arg.* $f(z)$ undergoes equal and opposite variations. It follows that if any plane area be divided into parts, as in Art. 111, the variation of the *arg.* $f(z)$ corresponding to the description in the same sense by z of all the partial areas, is equal to the variation of *arg.* $f(z)$ corresponding to the description by z of the external perimeter only. Now let any closed perimeter in the plane XY be described; and suppose in the first place, that it contains no point which satisfies the equation $f(z) = 0$. It can be broken up into a number of small areas, with respect to each of which the conclusions of (1) Art. 113 hold; and by what has been just proved it follows that the variation of *arg.* $f(z)$ corresponding to the description by z of the closed perimeter is nothing. Suppose in the second place, that the closed perimeter contains a point which is a root of the equation $f(z) = 0$ repeated m times. Let a small closed curve $PQRS$ be described round this point. The variation of *arg.* $f(z)$, corresponding to the description by z of the whole perimeter, is equal to the sum of its variations corresponding to the description of the areas $ABCPSR$, $CDARQP$, $PQRS$. The two former variations vanish by what is above proved; and the latter is, by (2), Art. 113, equal to $2m\pi$. The total variation, therefore, of $f(z)$ is $2m\pi$. Similarly, if the area includes a second, third, &c., points, which represent roots repeated m', m'', &c., times, the

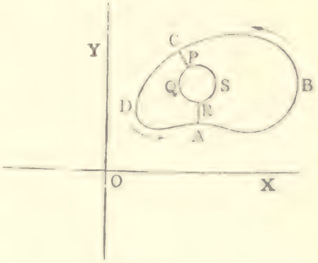

Fig. 11.

total variation $= 2(m + m' + m'' + \&c.)\,\pi$. Hence we derive the following theorem due to Cauchy:—

The number of roots of any polynomial, comprised within a given plane area, is obtained by dividing by 2π the total variation of the argument of this polynomial corresponding to the complete description by the complex variable of the perimeter of the area.

115. **Number of Roots of the General Equation.**—We are enabled by means of the principles established in the preceding Articles to prove the theorem contained in Arts. 15 and 16; namely, *Every rational and integral equation of the n^{th} degree has n roots real or imaginary.*

Let

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n$$

be a rational and integral function of z. Without making any supposition as to the existence of roots of $f(z) = 0$ further than that $f(z)$ cannot vanish for any infinite values of the variable, we can suppose z to describe in its plane a circle so large that no root exists outside of it. If, then,

$$f(z) = z^n \{a_0 + a_1 z' + a_2 z'^2 + \ldots + a^n z'^n\}$$

$$= z^n \phi(z'), \text{ where } z' = \frac{1}{z},$$

z', whose modulus is the reciprocal of the modulus of z, will describe a small circle containing a portion of the plane corresponding to the part outside of the circle described by z; and no root of $\phi(z') = 0$ will be included within this small circle. Hence, corresponding to the description of the whole circle by z, the variation of *arg.* $\phi(z') = 0$, and, therefore,

$$\text{variation of } arg.\, f(z) = \text{variation of } arg.\, z^n;$$

and if $\quad z = r(\cos\theta + i\sin\theta)$, or $z^n = r^n(\cos n\theta + i\sin n\theta)$,

θ is increased by 2π, and, therefore, *arg.* z^n is increased by $2n\pi$.

It follows from Cauchy's theorem, Art. 114, that the number of roots comprised within the circle described by z, i.e. the total

number of roots of the equation $f(z) = 0$, is n; and the theorem is proved.

The proposition whose proof was deferred in Art. 15 is thus shown to be an immediate consequence of Cauchy's theorem, which may therefore be regarded as the fundamental proposition of the Theory of Equations. It is proper to observe, however, that the theorem of Art. 15, viz., that every numerical equation has a numerical root, can be proved directly, and independently of Cauchy's theorem, by aid of the principles contained in Art. 112 and the preceding Articles, as we proceed now to show.

116. **Second Proof of Fundamental Theorem.**—If possible let there be no value of z which makes $f(z)$ vanish; and let the value z_0, represented by A, fig. 10, correspond to the nearest possible position, A', of P' to the origin O'. It is proposed to show that such a direction may be given to the increment h as to bring P' into a position nearer to the origin than A'. We have the following expansion (Art. 112):—

$$f(z_0 + h) = f(z_0) + f'(z_0)\,h + \frac{f''(z_0)}{1\,.\,2}\,h^2 + \ldots + a_0 h^n.$$

By hypothesis $f(z_0)$ does not vanish; but one or more of the derived functions, $f'(z_0)$, &c., may do so. Let the first of these which does not vanish be $f_m(z_0)$, and let us suppose

$$\frac{f_m(z_0)}{1\,.\,2\,.\,3\ldots m} = \mu_m(\cos a_m + i \sin a_m),$$

with corresponding expressions for the coefficients which follow. Collecting all the terms which contain powers of h beyond h^m into one complex expression, we may write

$$f(z_0 + h) = f(z_0) + \mu_m \rho^m \{\cos(m\phi + a_m) + i \sin(m\phi + a_m)\} + \mu(\cos\xi + i \sin\xi),$$

where, by the proposition of Art. 108,

$$\mu < \mu_{m+1}\rho^{m+1} + \mu_{m+2}\rho^{m+2} + \ldots + \mu_n \rho^n.$$

It is easily inferred from the theorem of Art. 5 that such a value may be given to ρ as to make $\mu < \mu_m \rho^m$. Now the direction of the increment h can be so selected, viz. from the equation $m\phi + a_m = X'O'A' + \pi$ (fig. 10), as to bring P', in virtue of the second expression in the value of $f(z_0 + h)$, through a distance $\mu_m \rho^m$ nearer to the origin in the direction $A'O'$. Let S be the point on the line $O'A'$ to which P' is brought in this way. The effect of the last expression in the value of $f(z_0 + h)$ is to move P' from S to a point T at a distance $ST = \mu$; and whatever the direction of this movement, *i.e.* whatever the argument ξ, $O'T$ is $< O'A'$, since $ST < SA'$. We have proved, therefore, that A' is not the nearest possible position of P' with reference to the origin; and in the same manner it may be shown that no other value different from zero can be the least possible value of the modulus of $f(z)$.

In the proof here given it is only shown that the equation must have a root, and the precise number of roots is not determined, as it is in the proof derived from Cauchy's Theorem; but when it is proved that one root at least must exist, the proof can be easily completed by the method of Art. 16.

It is important to observe that when $f'(z_0)$ does not vanish, for any particular point z_0 the limiting value of the ratio of the increment of $f(z_0)$ to h is the constant $f'(z_0) = \mu_1 (\cos a_1 + i \sin a_1)$. It is easily inferred that the two increments are inclined at a constant angle, and their moduli are in a constant ratio. This is usually expressed by saying that the figures described by P and P' are similar in their infinitely small parts.

The student is referred to Note C at the end of the volume for some further observations on the subject of this Article.

CHAPTER XII.

DETERMINANTS.

117. **Elementary Notions and Definitions.**—This chapter will be occupied with a discussion of an important class of functions which constantly present themselves in analysis. These functions possess remarkable properties, by a knowledge of which much simplification may be introduced into many mathematical operations.

The function $a_1 b_2 + a_2 b_1$, of the four quantities

$$\begin{matrix} a_1, & b_1, \\ a_2, & b_2, \end{matrix}$$

is obtained by assigning to a and b, written in alphabetical order, the suffixes 1, 2, and 2, 1, corresponding to the two permutations of the numbers 1, 2; and adding the two products so formed.

Similarly the function

$$a_1 b_2 c_3 + a_1 b_3 c_2 + a_2 b_3 c_1 + a_2 b_1 c_3 + a_3 b_1 c_2 + a_3 b_2 c_1, \qquad (1)$$

of the nine quantities

$$\begin{matrix} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \end{matrix}$$

is obtained by adding all the products abc which can be formed by assigning to the letters (retained in their alphabetical order) suffixes corresponding to all the permutations of the numbers 1, 2, 3. The whole expression might be represented by (abc), or any other convenient notation, from which all the terms could be written down.

The notation $(abcd)$ might be employed to represent a similar function of the 16 quantities a_1, b_1, c_1, d_1, a_2, &c., consisting of 24 terms, which can all be written down by the aid of the 24 permutations of the numbers 1, 2, 3, 4.

And, in general, taking n letters $a, b, c, \ldots l$, we can write down a similar function consisting of $n(n-1)(n-2)\ldots.3.2.1$ terms, this being the number of permutations of the first n numbers $1, 2, 3 \ldots n$.

Now the functions above referred to, which are of such frequent occurrence in mathematical analysis, differ from those just described in one respect only, namely: of the $1.2.3\ldots n$ (which is an even number) terms, half are affected with positive, and half with negative signs, instead of being all positive, as in the functions above given.

We shall now give some instances of the functions which will be discussed in this chapter. They occur most frequently as the result of elimination from linear equations. If, for example, x and y be eliminated from the equations

$$a_1x + b_1y = 0,$$

$$a_2x + b_2y = 0,$$

the result is $$a_1b_2 - a_2b_1 = 0.$$

Again, the result of eliminating x, y, z from the equations

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0,$$

is, as the student will readily perceive by solving from two of the equations and substituting in the third,

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 = 0; \qquad (2)$$

and this function differs from (1) written on the preceding page only in having three of its terms negative, instead of having the six terms positive.

Similarly the process of elimination from four linear equations gives rise to a function of the sixteen quantities

$$a_1,\ b_1,\ c_1,\ d_1,\ a_2,\ b_2,\ \&c.,$$

which differs from the function above represented by $(abcd)$ only in having twelve of its terms negative.

Expressions of the kind here described are called *Determinants.** The notation by which they are usually represented was first employed by Cauchy, and possesses many advantages in the treatment of these expressions. The quantities of which the function consists are arranged in a square between two vertical lines. For example, the notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

represents the determinant $a_1 b_2 - a_2 b_1$.

Similarly, the expression on the left-hand side of equation (2) is represented by the notation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

And, in general, the determinant of the n^2 quantities $a_1, b_1, c_1 \ldots l_1, a_2, b_2$, &c., is represented by

$$\begin{vmatrix} a_1 & b_1 & c_1 & . & . & . & l_1 \\ a_2 & b_2 & c_2 & . & . & . & l_2 \\ a_3 & b_3 & c_3 & . & . & . & l_3 \\ . & . & . & . & . & . & . \\ a_n & b_n & c_n & . & . & . & l_n \end{vmatrix}. \qquad (3)$$

By taking the n letters in alphabetical order, and assigning to them suffixes corresponding to the $n(n-1)(n-2) \ldots 3.\ 2.\ 1$ permutations of the numbers $1, 2, 3, \ldots n$, all the terms of the

* See Note D at the end of the volume.

determinant can be written down. Half of the terms must receive positive, and half negative signs. In the next Article the rule will be given by which the positive and negative terms are distinguished.

The individual letters $a_1, b_1, c_1, \ldots a_2, \ldots$ &c., of which a determinant is composed, are called *constituents*, and by some writers *elements*.

Any series of constituents such as $a_1, b_1, c_1, \ldots l_1$, arranged horizontally, form a *row* of the determinant; and any series such as $a_1, a_2, a_3, \ldots a_n$, arranged vertically, form a *column*.

The term *line* will sometimes be used to express a row or column indifferently.

118. **Rule with regard to Signs.**—It is evident from the preceding Article that each term of the determinant will, since it contains all the letters, contain one constituent (and only one) from every column; and will also, since the suffixes in each term comprise all the numbers, contain one constituent (and only one) from every row. We may thus regard the square array (3) of Art. 117 as the symbolical representation of a function consisting in general of $n(n-1)(n-2)\ldots 3.2.1$ terms, comprising all possible products which can be formed by taking one constituent, and one only, from each row; and one constituent, and one only, from each column. All that is required to give perfect definiteness to the function is to fix the sign to be attached to any particular term. For this purpose the following two rules are to be observed:—

(1). *The term $a_1 b_2 c_3 \ldots l_n$, formed by the constituents situated in the diagonal drawn from the left-hand top corner to the right-hand bottom corner is positive.*

This is called the *leading* or *principal* term. In it the suffixes and letters both occur in their natural order; and from it the sign of any other term is obtained by means of the following rule:—

(2). *Any interchange of two suffixes, the letters retaining their order, alters the sign of the term.*

This rule may be otherwise expressed thus :—*Any interchange of two letters, the suffixes retaining their order, alters the sign of a term.* For if two letters be interchanged, and the two corresponding constituents then interchanged, the entire process is equivalent to an interchange of suffixes. If, for example, in $a_1b_2c_3d_4e_5$ the letters b and e be interchanged, we get $a_1e_2c_3d_4b_5$, which is equal to $a_1b_5c_3d_4e_2$, and this is derived from the given term by an interchange of the suffixes 2 and 5.

In applying this rule it is evident that an even number of interchanges will not alter the sign of a term, and that an odd number will.

Examples.

1. What sign is to be attached to the term $a_3b_4c_2d_5e_1$ in the determinant of the 5th order?

The question is, How many interchanges will change the order 12345 into 34251? Here, when 3 is interchanged with 2, and afterwards with 1, it comes into the leading place, the order becoming 31245. Again, the interchange in 31245 of 4 with 2, and afterwards with 1, presents the order 34125. The interchange of 2 with 1 gives the order 34215; and finally the interchange of 5 with 1 gives the required order 34251. Thus there are in all six interchanges; and therefore the required sign is positive.

The general mode of proceeding may plainly be stated as follows :—Take the figure which stands first in the required order, and move it from its place in the natural order 1234 . . . into the leading place, counting one displacement for each figure passed over. Take then the figure which stands second in the required order, and move it from its place in the natural order into the second place; and so on. If the number of displacements in this process be even, the sign is positive; if it be odd, the sign is negative.

2. What sign is to be attached to the term $a_3b_7c_6d_5e_1f_4g_2$ in the determinant of the 7th order?

Here two displacements bring 3 to the leading place; five displacements then bring 7 to the second place; four then bring 6 to the third place; three then bring 5 to the fourth place; the figure 1 is in its place; and finally, one displacement brings 4 into the sixth place. Thus there are in all fifteen displacements; and the required sign is therefore negative.

3. Write down all the terms of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

The six permutations of suffixes in which the figure 1 occurs first are

1234, 1243, 1324, 1342, 1423, 1432.

The six corresponding terms are, as the student will easily see by applying the Rule (2), as in the previous examples,

$$a_1b_2c_3d_4 - a_1b_2c_4d_3 + a_1b_3c_4d_2 - a_1b_3c_2d_4 + a_1b_4c_2d_3 - a_1b_4c_3d_2.$$

The other eighteen terms, corresponding to the permutations in which the figures 2, 3, 4, respectively, stand first, are as follows:—

$$\begin{aligned} &a_2b_1c_4d_3 - a_2b_1c_3d_4 + a_2b_3c_1d_4 - a_2b_3c_4d_1 + a_2b_4c_3d_1 - a_2b_4c_1d_3 \\ &+ a_3b_1c_2d_4 - a_3b_1c_4d_2 + a_3b_2c_4d_1 - a_3b_2c_1d_4 + a_3b_4c_1d_2 - a_3b_4c_2d_1 \\ &+ a_4b_1c_3d_2 - a_4b_1c_2d_3 + a_4b_2c_1d_3 - a_4b_2c_3d_1 + a_4b_3c_2d_1 - a_4b_3c_1d_2. \end{aligned}$$

It will be observed here that the number of positive terms is equal to the number of negative terms. The same must be true in general; for, corresponding to any positive term there exists a negative term obtained by simply interchanging the last two suffixes.

4. Show that if any two adjacent figures be moved together over any number m of figures, the sign is unaltered.

For if they be moved separately, the whole process is equivalent to a movement over $2m$ figures.

5. Determine the sign to be attached to the second diagonal term, viz. $a_nb_{n-1}c_{n-2} \ldots k_2l_1$, in the determinant of the n^{th} order.

Here the number of displacements required to change the natural order to the required order is plainly

$$(n-1) + (n-2) + (n-3) + \ldots + 2 + 1 = \frac{n(n-1)}{2}.$$

Hence the required sign is $(-1)^{\frac{n(n-1)}{2}}$.

119. In the Propositions of the present and following Articles are contained the most important elementary properties of determinants which, by the aid of Cauchy's notation above described, render the employment of these functions of such practical advantage.

Prop. I.—*If any two rows, or any two columns, of a determinant be interchanged, the sign of the determinant is changed.*

This follows at once from the mode of formation (Rule (2), Art. 118), for an interchange of two rows is the same as an interchange of two suffixes, and an interchange of two columns is the same as an interchange of two letters; so that in either case the sign of every term of the determinant is changed.

By aid of this proposition the rule for obtaining the sign of any term may be stated in a form which is usually more convenient for practical purposes than that already given. It will readily be perceived that the general mode of procedure explained in Ex. 1, Art. 118, is equivalent to the following:—*Bring, by movements of rows (or columns), the constituents of the term whose sign is required into the position of the leading diagonal. The sign of the term will be positive or negative according as the number of displacements is even or odd.*

EXAMPLE.

What sign is to be attached to the term $\lambda\beta n x$ in the determinant

$$\begin{vmatrix} a & b & c & \underline{x} \\ \alpha & \underline{\beta} & \gamma & y \\ l & m & \underline{n} & z \\ \underline{\lambda} & \mu & \nu & 0 \end{vmatrix}?$$

Here a movement of the fourth row over three rows (*i. e.* three displacements) brings λ into the leading place. One displacement of the original second row upwards brings β into the required place in the diagonal term. And one further displacement of the original third row upwards effects the required arrangement, bringing $\lambda\beta n x$ into the diagonal place. Thus the number of displacements being odd, the required sign is negative.

120. PROP. II.—*Whenever, in any determinant, two rows or two columns are identical, the determinant vanishes.*

For, by Prop. I., the interchange of these two lines ought to change the sign of the determinant Δ; but the interchange of two identical rows or columns cannot alter the determinant in any way; hence $\Delta = -\Delta$, or $\Delta = 0$.

121. PROP. III.—*The value of a determinant is not altered if the rows be written as columns, and the columns as rows.*

For all the terms, formed by taking one constituent from each row and one from each column, are plainly the same in value in both cases; the principal term is identically the same; and to determine the sign of any other term (by Prop. I.) the

number of displacements of rows necessary to bring it into the leading diagonal in the first case is the same as the number of displacements of columns necessary in the second case.

EXAMPLE.

$$\begin{vmatrix} a_1 & b_1 & \underline{c_1} & d_1 \\ \underline{a_2} & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & \underline{d_3} \\ a_4 & \underline{b_4} & c_4 & d_4 \end{vmatrix} \equiv \begin{vmatrix} a_1 & \underline{a_2} & a_3 & a_4 \\ b_1 & b_2 & b_3 & \underline{b_4} \\ \underline{c_1} & c_2 & c_3 & c_4 \\ d_1 & d_2 & \underline{d_3} & d_4 \end{vmatrix}.$$

Here the sign of any term, e.g. $a_2b_4c_1d_3$, is the same in both determinants. For three displacements of rows are required to bring this term into the leading position in the first determinant; and the same number of displacements of columns is required to bring the same constituents into the leading position in the second determinant.

122. PROP. IV.—*If every constituent in any line be multiplied by the same factor, the determinant is multiplied by that factor.*

For every term of the determinant must contain one, and only one, constituent from any row or any column.

Cor. 1. If the constituents in any line differ only by the same factor from the constituents in any parallel line, the determinant vanishes.

Cor. 2. If the signs of all the constituents in any line be changed, the sign of the determinant is changed. For this is equivalent to multiplying by the factor -1.

EXAMPLES.

1. $$\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} \equiv k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2. $$\begin{vmatrix} \alpha_1 & m\alpha_1 & \alpha_2 \\ \beta_1 & m\beta_1 & \beta_2 \\ \gamma_1 & m\gamma_1 & \gamma_2 \end{vmatrix} \equiv m \begin{vmatrix} \alpha_1 & \alpha_1 & \alpha_2 \\ \beta_1 & \beta_1 & \beta_2 \\ \gamma_1 & \gamma_1 & \gamma_2 \end{vmatrix} \equiv 0.$$

3. Show that the following determinant vanishes:—

$$\begin{vmatrix} 3 & 1 & 5 & 2 \\ 2 & 5 & 7 & 3 \\ 8 & 9 & 1 & 4 \\ 6 & 15 & 21 & 9 \end{vmatrix}.$$

When the constituents of the last row are divided by 3, they become identical with those of the second row.

4. Prove the identity

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Represent the first determinant by Δ, and multiply the rows by a, b, c, respectively. We have then

$$abc\,\Delta = \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix};$$

and, dividing the first column by abc, the result follows.

5. Prove the identity

$$\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}.$$

6. Prove

$$\begin{vmatrix} 2 & 1 & -7 \\ -4 & -3 & 8 \\ 6 & 5 & -9 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 8 \\ 3 & 5 & 9 \end{vmatrix}.$$

Change all the signs of the second row, and afterwards of the third column.

7. Prove

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \frac{1}{\alpha\beta\gamma}\begin{vmatrix} 1 & 1 & 1 \\ \alpha'\beta\gamma & \beta'\gamma\alpha & \gamma'\alpha\beta \\ \alpha''\beta\gamma & \beta''\gamma\alpha & \gamma''\alpha\beta \end{vmatrix}.$$

This is easily proved by multiplying the columns of the first determinant by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, respectively; and then dividing the first row by $\alpha\beta\gamma$.

It is evident that a similar process may be employed in general to reduce any determinant to one in which all the constituents of any selected row or column shall be units.

8. Reduce the following determinant to one in which the first row shall consist of units:—

$$\Delta \equiv \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}.$$

Since 20 is the least common multiple of 4, 2, 5, 10, it is sufficient to multiply the columns in order by 5, 10, 4, 2; we thus obtain

$$\Delta = \frac{1}{5 \cdot 10 \cdot 4 \cdot 2} \begin{vmatrix} 20 & 20 & 20 & 20 \\ 5 & 10 & 24 & 6 \\ 35 & 30 & 0 & 10 \\ 0 & 20 & 20 & 16 \end{vmatrix}.$$

Taking out the multiplier 20 from the first row, 5 from the third row, and 4 from the fourth row, we get finally

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 10 & 24 & 6 \\ 7 & 6 & 0 & 2 \\ 0 & 5 & 5 & 4 \end{vmatrix}.$$

9. Prove the identity

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \equiv (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

Since if β were equal to γ, two columns would become identical, $\beta - \gamma$ must be a factor in the determinant. Similarly, $\gamma - \alpha$ and $\alpha - \beta$ must be factors in it. Hence the product of the three differences can differ by a numerical factor only from the value of the determinant, since both functions are of the third degree in α, β, γ; and by comparing the term $\beta\gamma^2$ we observe that this factor is $+ 1$.

10. Prove similarly the identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \equiv -(\beta - \gamma)(\alpha - \delta)(\gamma - \alpha)(\beta - \delta)(\alpha - \beta)(\gamma - \delta).$$

It is evident that a similar proof shows in general that the value of the determinant of this form, constituted by the n quantities $\alpha, \beta, \gamma \ldots \lambda$, is the product of the $\frac{1}{2}n(n-1)$ differences which can be formed with these n quantities.

123. **Minor Determinants. Definitions.**—When in a determinant any number of rows, and the same number of columns, are suppressed, the determinant formed by the remaining constituents (maintaining their relative positions) is called a *minor determinant.*

If one row and one column only be suppressed, the corresponding minor is called a *first minor.* If two rows and two columns be suppressed, the minor is called a *second minor;* and so on. The suppressed rows and columns have common constituents forming a determinant; and the minor which remains is said to be *complementary* to this determinant. The minor complementary to the leading constituent a_1 is called the *leading first minor*, and its leading first minor again is the *leading second minor* of the original determinant.

It is usual to denote a determinant in general by Δ. We shall denote by Δ_a the first minor obtained by suppressing in Δ the row and column which contain any constituent a; by $\Delta_{a,\beta}$ the second minor obtained by suppressing the two rows and two columns which contain a and β; and so on. Thus Δ_{a_1} represents the leading first minor, and Δ_{a_1,b_2} or Δ_{a_2,b_1} the leading second minor.

The determinant Δ, formed by the constituents a_1, b_1, c_1, &c., is often denoted for brevity by placing the leading term within brackets, as follows: $\Delta = (a_1 b_2 c_3 \ldots\ldots l_n)$. The notation $\Sigma \pm a_1 b_2 c_3 \ldots l_n$ is also used to represent Δ; this expressing its constitution as consisting of the sum of a number of terms (with their proper signs attached) formed by taking all possible permutations of the n suffixes.

124. **Development of Determinants.**—Since every term of any determinant contains one, and only one, constituent from each row and from each column, it follows that Δ *is a linear and homogeneous function of the constituents of any one row or any one*

column. Thus we may write

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 + \&c.,$$
$$\Delta = b_1 B_1 + b_2 B_2 + b_3 B_3 + \&c.;$$

or, again,

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1 + \&c.,$$
$$\Delta = a_2 A_2 + b_2 B_2 + c_2 C_2 + \&c.$$

The student, on referring to Ex. 3, Art. 118, will observe that the determinant of the fourth order there written at length is constituted in the way here described, namely,

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_4 & c_4 & d_4 \\ b_3 & c_3 & d_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_2 & c_2 & d_2 \end{vmatrix}.$$

We proceed to show that in the general case, writing Δ in the form

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 + \ldots + a_n A_n,$$

the coefficients A_1, A_2, A_3, &c., are determinants of the order $n-1$.

In effecting all the permutations of the suffixes $1, 2, 3 \ldots . n$, suppose first 1 to remain in the leading place, as in the example referred to; we then obtain $1 . 2 . 3 \ldots . (n-1)$ terms which have a_1 as a factor, and

$$a_1 A_1 = a_1 \Sigma \pm b_2 c_3 \ldots l_n;$$

hence

$$A_1 = \Sigma \pm b_2 c_3 \ldots l_n = \begin{vmatrix} b_2 & c_2 \ldots l_2 \\ b_3 & c_3 \ldots l_3 \\ \cdot & \cdot \quad \cdot \quad \cdot \\ b_n & c_n \ldots l_n \end{vmatrix};$$

and this determinant is the minor corresponding to the constituent a_1, or $A_1 = \Delta_{a_1}$.

To find the value of A_2, we bring a_2 into the leading place by one displacement of rows. This changes the sign of Δ, so that we obtain $A_2 = -\Delta_{a_2}$, i.e. A_2 = the minor corresponding to a_2 with its side changed. Again, bringing a_3 to the leading place by two displacements, we have $A_3 = \Delta_{a_3}$; and so on.

Thus we conclude that, in general,

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4} + \&c.$$

Similarly, we can expand Δ in terms of the constituents of any other column, or any row. For example,

$$\Delta = a_1\Delta_{a_1} - b_1\Delta_{b_1} + c_1\Delta_{c_1} - \&c.$$

If it be required to obtain the proper sign to be attached to the minor which multiplies any constituent in the expanded form, we have only to consider how many displacements would bring that constituent to the leading place. For example, suppose the determinant $(a_1b_2c_3d_4e_5)$ is expanded in terms of its fourth column, and that it is required to find what sign is to be attached to $d_3\Delta_{d_3}$. Here two displacements upwards, and afterwards three to the left, will bring d_3 to the leading place; hence the sign is negative. This rule may be stated simply as follows:— *Proceed from a_1 to the constituent under consideration along the top row, and down the column containing the constituent; the number of letters passed over before reaching the constituent will decide the sign to be attached to the minor.* In the example just given, beginning at a_1, we count a_1, b_1, c_1, d_1, d_2, i.e. five; and this number being odd, the required sign is negative.

It will be found convenient to retain both notations here employed for the development of a determinant. The expansion in terms of the minors, with signs alternately positive and negative, is useful in calculating the value of a determinant by successive reductions to determinants of lower degree. For some purposes, as will appear in the Articles which follow, it is more convenient to employ the notation first given, in which the signs are all positive (whatever the row or column under consideration), and the coefficient of any constituent represented by the corresponding capital letter. By substituting for the capital letter the corresponding minor with the proper sign, determined in the manner above explained, the latter notation is changed into the former.

Examples.

1. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

(Compare (2), Art. 117.)

2. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \begin{vmatrix} h & g \\ b & f \end{vmatrix}$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$

3. Expand the determinant of the fourth order in terms of the constituents of the fourth row.

$$\Delta = -a_4\Delta_{a_4} + b_4\Delta_{b_4} - c_4\Delta_{c_4} + d_4\Delta_{d_4}$$

$$= -a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_4 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

When the determinants of the third order are expanded this will give the expression of Ex. 3, Art. 118, as the student will easily verify.

4. $$\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix} = 3 \begin{vmatrix} 6 & 1 \\ 3 & 8 \end{vmatrix} - 7 \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix}$$

$$= 3\,(48 - 3) - 7\,(16 - 12) + 5\,(2 - 24)$$
$$= -3.$$

5. Find the value of the determinant

$$\Delta = \begin{vmatrix} 8 & 7 & 2 & 20 \\ 3 & 1 & 4 & 7 \\ 5 & 0 & 11 & 0 \\ 8 & 1 & 0 & 6 \end{vmatrix}.$$

It is plainly convenient to expand this in terms of the third row, since two of the constituents in that row vanish.

$$\Delta = 5 \begin{vmatrix} 7 & 2 & 20 \\ 1 & 4 & 7 \\ 1 & 0 & 6 \end{vmatrix} + 11 \begin{vmatrix} 8 & 7 & 20 \\ 3 & 1 & 7 \\ 8 & 1 & 6 \end{vmatrix};$$

and expanding the two determinants of the third order, we find $\Delta = 2188$.

6. Expand
$$\begin{vmatrix} 0 & c & b & d \\ c & 0 & a & e \\ b & a & 0 & f \\ d & e & f & 0 \end{vmatrix}.$$

Ans. $a^2d^2 + b^2e^2 + c^2f^2 - 2bcef - 2cafd - 2abde.$

7. Prove
$$\begin{vmatrix} 1 & \alpha & \beta & \gamma \\ -\alpha & 1 & \gamma' & -\beta' \\ -\beta & -\gamma' & 1 & \alpha' \\ -\gamma & \beta' & -\alpha' & 1 \end{vmatrix} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \alpha'^2 + \beta'^2 + \gamma'^2 + (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2.$$

8. Expand
$$\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}.$$

Ans. $a^4 + b^4 + c^4 + d^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 - 8abcd.$

9. Prove the following identity, and expand the determinants:—
$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

Ans. $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2.$

10. Find the value of the determinant
$$\Delta \equiv \begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix}.$$

Expand first in terms of the last row or last column, and then each of the determinants of the third order in terms of λ, μ, ν.

Ans. $-\Delta = (bc - f^2)\lambda^2 + (ca - g^2)\mu^2 + (ab - h^2)\nu^2 + 2(gh - af)\mu\nu$
$+ 2(hf - bg)\nu\lambda + 2(fg - ch)\lambda\mu.$

125. **Laplace's Development of a Determinant.**—The expansion explained in the preceding Article is included in a more general mode of development given by Laplace. In place of expanding the determinant as a linear function of the constituents of any line, we now expand it as a linear function of the minors comprised in any number of lines.

Consider, for example, the first two columns (a, b) of any determinant; and let all possible determinants of the second order $(a_p b_q)$, obtained by taking any two rows of these two columns, be formed. Let the minor formed by suppressing the a_p and b_q lines be represented by $\Delta_{p,q}$; then the determinant can be expanded in the form $\Sigma \pm (a_p b_q) \Delta_{p,q}$, where each term is the product of two complementary determinants (see Art. 123). To prove this, we observe that every term of the determinant must contain one constituent from the column a and one from the column b. Suppose a term to contain the factor $a_p b_q$, there must then (interchanging p and q) be another term differing only in the sign and the interchange of these suffixes; hence, the determinant can be expanded in the form $\Sigma (a_p b_q) A_{p,q}$; and $A_{p,q}$ is plainly the sum of all the terms which can be obtained by permuting in every possible way the $n - 2$ suffixes of the letters c, d, e, &c., viz. $\pm \Delta_{p,q}$, the sign being determined in any particular instance by the rule of Art. 118. This reasoning can easily be extended to the general case. Let any number p of columns be taken, and all possible minors formed by taking p rows of these columns. Each of these minors is to be then multiplied by the complementary minor, and the determinant expressed as the sum of all such products with their proper signs.

Examples.

1. Expand the determinant $(a_1 b_2 c_3 d_4)$ in terms of the minors of the second order formed from the first two columns.

Employing the bracket notation, we can write down the result as follows:—

$$(a_1 b_2)(c_3 d_4) - (a_1 b_3)(c_2 d_4) + (a_1 b_4)(c_2 d_3) + (a_2 b_3)(c_1 d_4) - (a_2 b_4)(c_1 d_3) + (a_3 b_4)(c_1 d_2);$$

where the sign to be attached to any product is determined by moving the two rows involved in the first factor into the positions of first and second row. Thus, for

example, since three displacements are required to move the second and fourth rows into these positions, the sign of the product $(a_2b_4)(c_1d_3)$ is negative.

2. Expand similarly the determinant $(a_1b_2c_3d_4e_5)$.

$$\begin{aligned} \textit{Ans. } & (a_1b_2)(c_3d_4e_5) - (a_1b_3)(c_2d_4e_5) + (a_1b_4)(c_2d_3e_5) - (a_1b_5)(c_2d_3e_4) \\ & + (a_2b_3)(c_1d_4e_5) - (a_2b_4)(c_1d_3e_5) + (a_2b_5)(c_1d_3e_4) + (a_3b_4)(c_1d_2e_5) \\ & - (a_3b_5)(c_1d_2e_4) + (a_4b_5)(c_1d_2e_3). \end{aligned}$$

3. Prove the identity

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

This appears by expanding the determinant in terms of the minors formed from the first three columns, for it is evident that all these minors vanish (having one row at least of ciphers) except one, viz. $(a_1b_2c_3)$.

In general it appears in the same way that if a determinant of the $2m^{th}$ order contains in any position a square of m^2 ciphers, it can be expressed as the product of two determinants of the m^{th} order.

4. Expand the determinant

$$\begin{vmatrix} a & h & g & \lambda & \lambda' \\ h & b & f & \mu & \mu' \\ g & f & c & \nu & \nu' \\ \lambda & \mu & \nu & 0 & 0 \\ \lambda' & \mu' & \nu' & 0 & 0 \end{vmatrix}$$

in powers of α, β, γ, where

$$\alpha \equiv \mu\nu' - \mu'\nu, \quad \beta \equiv \nu\lambda' - \nu'\lambda, \quad \gamma \equiv \lambda\mu' - \lambda'\mu.$$

Ans. $a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta$.

5. Verify the development of the present Article by showing that it gives in the general case the proper number of terms.

Consider the first r columns of a determinant of the n^{th} order. The number of minors formed from these is equal to the number of combinations of n things taken r together. This number multiplied by $1 . 2 . 3 \ldots r$ (the number of terms in each minor), and $1 . 2 . 3 \ldots n - r$ (the number of terms in each complementary minor), will be found to give $1 . 2 . 3 \ldots n$, viz. the number of terms in the determinant.

126. **Development of a Determinant in Products of the leading Constituents.**—In this and the next following Articles will be explained two additional modes of development which will be found useful in the expansion of certain determinants of special form. The application which follows will be sufficient to show how any determinant may be expanded in products of the leading constituents—

It is required to expand the determinant of the fourth order

$$\Delta \equiv \begin{vmatrix} A & b_1 & c_1 & d_1 \\ a_2 & B & c_2 & d_2 \\ a_3 & b_3 & C & d_3 \\ a_4 & b_4 & c_4 & D \end{vmatrix}$$

according to the products of A, B, C, D. In order to give prominence to the leading constituents we have here replaced a_1, b_2, c_3, d_4 by A, B, C, D. When the expansion is effected it is plain that the result must be of the form

$$\Delta \equiv \Delta_0 + \Sigma\lambda A + \Sigma\lambda' AB + ABCD,$$

where Δ_0 consists of all the terms in which no leading constituent occurs; $\Sigma\lambda A$ is the sum of all the terms in which one only of these constituents occurs; $\Sigma\lambda' AB$ is the sum of all in which the product of a pair of the leading constituents is found; and $ABCD$, the leading term, is the product of all these constituents. It will be observed that the expansion here written contains no terms of the form $\lambda'' ABC$, and it is evident in general that the expanded determinant can contain no terms in which products of all the leading constituents but one occur, since the coefficient of any such product is the remaining diagonal constituent. It only remains to see what is the form of Δ_0, and of the undetermined coefficients λ, μ, . . . λ', μ', . . . &c.

Putting A, B, C, D all equal to zero in the identity above written, we have

$$\Delta_0 = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}.$$

Again, to obtain λ, let B, C, D be made equal to zero. The coefficient of A is plainly the determinant

$$\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix};$$

the coefficient of B is similarly obtained by replacing A, C, D each by zero in the

minor complementary to B; and so on. To obtain λ', let C and D be made zero: the coefficient of AB in the resulting determinant is plainly the second minor

$$\begin{vmatrix} 0 & d_3 \\ c_4 & 0 \end{vmatrix}.$$

The coefficient of any other product is obtained in a similar manner. Finally, the expansion of Δ may be written in the form

$$\begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}$$

$$+ A\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix} + B\begin{vmatrix} 0 & c_1 & d_1 \\ a_3 & 0 & d_3 \\ a_4 & c_4 & 0 \end{vmatrix} + C\begin{vmatrix} 0 & b_1 & d_1 \\ a_2 & 0 & d_2 \\ a_4 & b_4 & 0 \end{vmatrix} + D\begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & b_3 & 0 \end{vmatrix}$$

$$+ AB\begin{vmatrix} 0 & d_3 \\ c_4 & 0 \end{vmatrix} + AC\begin{vmatrix} 0 & d_2 \\ b_4 & 0 \end{vmatrix} + AD\begin{vmatrix} 0 & c_2 \\ b_3 & 0 \end{vmatrix} + BC\begin{vmatrix} 0 & d_1 \\ a_4 & 0 \end{vmatrix} + BD\begin{vmatrix} 0 & c_1 \\ a_3 & 0 \end{vmatrix} + CD\begin{vmatrix} 0 & b_1 \\ a_2 & 0 \end{vmatrix}$$

$$+ ABCD.$$

A determinant whose leading constituents all vanish has been called *zero-axial.* The result just obtained may be stated as follows :—*Any determinant may be expanded in products of the leading constituents, the coefficient of every product in the result being a zero-axial determinant.*

127. **Expansion of a Determinant by Products in Pairs of the Constituents of a Row and Column.**—In what follows we take the first row and first column as those in terms of which the expansion is required. This is plainly sufficient, since any other row and column may be brought by displacements into these positions. It will be found convenient to write the determinant under consideration in the form

$$\begin{vmatrix} a_0 & a & \beta & \gamma & . \\ a' & a_1 & b_1 & c_1 & . \\ \beta' & a_2 & b_2 & c_2 & . \\ \gamma' & a_3 & b_3 & c_3 & . \\ . & . & . & . & . \end{vmatrix}.$$

Let this be denoted by Δ', and its leading first minor $(a_1 b_2 c_3 \ldots)$ by the usual notation Δ. The determinant Δ' may be said to be derived from Δ by *bordering* it, horizontally with the constituents a_0, a, β, γ, . . ., and vertically with the constituents a_0, a', β', γ', . . . When Δ' is expanded, all the terms which contain a_0 are included in $a_0\Delta$. In addition to this, the expansion will consist of the product of every other constituent of the first column by every other constituent of the first row, every such product of two being multiplied by its proper factor. What this factor is in the case of any product is easily seen. Let the coefficients of a_1, b_1, c_1, . . . a_2, b_2, . . . &c., in the expansion of Δ be A_1, B_1, . . . A_2, B_2, . . . , according to the notation explained in Art. 124. It is plain that the factor which multiplies any product, for example aa', in the expansion of Δ', is the same as the factor which multiplies $a_0 a_1$ with sign changed, viz. $-A_1$; similarly the factor which multiplies $a'\beta$ is the factor with sign changed of $a_0 b_1$, viz. $-B_1$; and so on. To obtain the factor of any such product the rule plainly is—*Find the fourth constituent completing the rectangle formed by the leading term a_0 and the two constituents which enter into the product: the required factor is obtained by substituting for the constituent of Δ so found the corresponding capital letter with the negative sign.* It appears therefore finally that the expansion of Δ' may be written in the following form:—

$$\begin{aligned}\Delta' = a_0\Delta &- A_1 aa' - B_1\beta a' - C_1\gamma a' - \ldots \\ &- A_2 a\beta' - B_2\beta\beta' - C_2\gamma\beta' - \ldots \\ &- A_3 a\gamma' - B_3\beta\gamma' - C_3\gamma\gamma' - \ldots \\ &- \&c.\end{aligned}$$

Examples of the utility of this mode of expansion will be found under a subsequent Article.

128. **Addition of Determinants.** PROP. V.—*If every constituent in any line can be resolved into the sum of two others, the determinant can be resolved into the sum of two others.*

Suppose the constituents of the first column to be $a_1 + \alpha_1$, $a_2 + \alpha_2$, $a_3 + \alpha_3$, &c. Substituting these in the expansion of Art. 124, we have

$$\Delta = (a_1 + \alpha_1) A_1 + (a_2 + \alpha_2) A_2 + (a_3 + \alpha_3) A_3 + \&c.$$

$$= a_1 A_1 + a_2 A_2 + a_3 A_3 + \ldots \&c. + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \&c.\,;$$

or,

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 & \ldots \\ a_2 + \alpha_2 & b_2 & c_2 & \ldots \\ a_3 + \alpha_3 & b_3 & c_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ a_3 & b_3 & c_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 & \ldots \\ \alpha_2 & b_2 & c_2 & \ldots \\ \alpha_3 & b_3 & c_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

which proves the proposition.

If a second column consists of the sum of two others, it is easily seen, by first resolving with reference to one column, and afterwards with reference to the other, that the determinant can be resolved into the sum of four others. For example, the determinant

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 \\ a_2 + \alpha_2 & b_2 + \beta_2 & c_2 \\ a_3 + \alpha_3 & b_3 + \beta_3 & c_3 \end{vmatrix}$$

is (using the notation of Art. 123) equal to the sum of the four determinants

$$(a_1 b_2 c_3) + (\alpha_1 b_2 c_3) + (a_1 \beta_2 c_3) + (\alpha_1 \beta_2 c_3).$$

Similarly it follows that if each of the constituents of one column consists of the algebraical sum of any number of terms, the determinant can be resolved into a corresponding number of determinants. For example—

$$\begin{vmatrix} a_1 - \alpha_1 + \alpha'_1 & b_1 & c_1 \\ a_2 - \alpha_2 + \alpha'_2 & b_2 & c_2 \\ a_3 - \alpha_3 + \alpha'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha'_1 & b_1 & c_1 \\ \alpha'_2 & b_2 & c_2 \\ \alpha'_3 & b_3 & c_3 \end{vmatrix}.$$

And, in general, if one column consists of the algebraic sum of m others, a second column of the sum of n others, a third of the sum of p others, &c., the determinant can be resolved into the sum of $mnp \ldots$, &c., others.

Similar results plainly hold with regard to the rows, which may be substituted for columns in the proof just given.

129. Prop. VI.—*If the constituents of one line are equal to the sums of the corresponding constituents of the other lines multiplied by constant factors, the determinant vanishes.*

For it can then be resolved into the sum of a number of determinants which separately vanish. For example,

$$\begin{vmatrix} ma_1 + nb_1 & a_1 & b_1 \\ ma_2 + nb_2 & a_2 & b_2 \\ ma_3 + nb_3 & a_3 & b_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} + n \begin{vmatrix} b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \\ b_3 & a_3 & b_3 \end{vmatrix},$$

and each of the latter determinants vanishes (Art. 120).

130. Prop. VII.—*A determinant is unchanged when to each constituent of any row or column are added those of several other rows or columns, multiplied respectively by constant factors.*

For when the determinant is resolved into the sum of others, as in Art. 128, the determinants in which the added lines occur all vanish, since each of them must, when the constant factor is removed, contain two identical lines. Thus, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix};$$

for when the second determinant is expressed as the sum of three others, the two arising from the added columns vanish identically (Art. 129).

The proposition of the present Article supplies in practice one of the most useful properties in the evaluation of determinants.

Examples.

1. Show that the following determinant vanishes :—

$$\begin{vmatrix} \beta+\gamma & \alpha & 1 \\ \gamma+\alpha & \beta & 1 \\ \alpha+\beta & \gamma & 1 \end{vmatrix}.$$

Adding the constituents of the second column to those of the first, we can take out $\alpha+\beta+\gamma$ as a factor, and two columns then become identical.

2. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{vmatrix}.$$

Subtracting the constituents of the first column from those of the second, and three times the constituents of the first column from those of the third, we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix},$$

which vanishes identically.

3. $$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -16.$$

Here the first transformation is obtained by adding in succession the constituents of the first row to those of the second, third, and fourth.

4. $$\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 3 & 9 & 6 \end{vmatrix} = 3\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 1 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 7 & -10 & -10 \\ 13 & -24 & -16 \\ 1 & 0 & 0 \end{vmatrix} = 3\begin{vmatrix} 10 & 10 \\ 24 & 16 \end{vmatrix}$$

$$= 30\,(16-24) = -240.$$

Here the second transformation is obtained by subtracting three times the first column from the second, and twice the first from the third. In examples of this kind attempts should be made to reduce to zero all the constituents except one in some row or column, in which case the calculation reduces to that of a determinant of lower order. This can always be done by reducing any one line to units, as

in Ex. 7, Art. 122; but in general it can be effected more readily by direct additions or subtractions, as in the present instance.

5. $$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -2 & 0 & 5 \\ 19 & 0 & -2 & 17 \\ -7 & 0 & 5 & -2 \\ 12 & 0 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 19 & -2 & 17 \\ -7 & 5 & -2 \\ 12 & 3 & 9 \end{vmatrix}.$$

The first transformation is obtained by adding to the second row three times the first, subtracting the first from the third row, and adding the first to the fourth row. The reduced determinant is easily calculated by subtracting four times the second column from the first, and three times the second column from the third. Thus

$$2 \begin{vmatrix} 19 & -2 & 17 \\ -7 & 5 & -2 \\ 12 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 27 & -2 & 23 \\ -27 & 5 & -17 \\ 0 & 3 & 0 \end{vmatrix} = -6 \begin{vmatrix} 27 & 23 \\ -27 & -17 \end{vmatrix} = -972.$$

6. Calculate the determinant

$$\Delta = \begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}.$$

The first sixteen natural numbers are arranged here in what is called a "magic square," *i.e.* the sum of all the figures in any row or in any column is constant. In general for a square of the first n^2 numbers this sum is $\frac{1}{2}n(n^2+1)$. Determinants of this kind can be at once reduced one degree. Here, adding the last three columns to the first, and subtracting the last row from each of the others, we have

$$\Delta = 34 \begin{vmatrix} 1 & 15 & 14 & 4 \\ 1 & 6 & 7 & 9 \\ 1 & 10 & 11 & 5 \\ 1 & 3 & 2 & 16 \end{vmatrix} = 34 \begin{vmatrix} 0 & 12 & 12 & -12 \\ 0 & 3 & 5 & -7 \\ 0 & 7 & 9 & -11 \\ 1 & 3 & 2 & 16 \end{vmatrix} = -34 \times 12 \begin{vmatrix} 1 & 1 & -1 \\ 3 & 5 & -7 \\ 7 & 9 & -11 \end{vmatrix};$$

and subtracting the second row from the last row, it is evident that the reduced determinant vanishes; hence $\Delta = 0$.

7. Calculate the determinant formed by the first nine natural numbers arranged in a magic square:

$$\begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}.$$

Ans. 360.

8. Calculate the determinant formed by the first twenty-five natural numbers arranged in a magic square:

$$\begin{vmatrix} 10 & 18 & 1 & 14 & 22 \\ 4 & 12 & 25 & 8 & 16 \\ 23 & 6 & 19 & 2 & 15 \\ 17 & 5 & 13 & 21 & 9 \\ 11 & 24 & 7 & 20 & 3 \end{vmatrix}.$$

Ans. -4680000.

9. Evaluate, by the method of the present Article, the determinant of Ex. 9, Art. 124.

$$\Delta = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & -z^2 & x^2-z^2 \\ 1 & y^2 & x^2-y^2 & -y^2 \end{vmatrix} = -\begin{vmatrix} 1 & z^2 & y^2 \\ 1 & -z^2 & x^2-z^2 \\ 1 & x^2-y^2 & -y^2 \end{vmatrix}.$$

Here, to obtain the second determinant, we subtract the second column from each of the following ones. In the reduced determinant, subtracting the first row from each of the following, we find

$$\Delta = -\begin{vmatrix} 1 & z^2 & y^2 \\ 0 & -2z^2 & x^2-z^2-y^2 \\ 0 & x^2-y^2-z^2 & -2y^2 \end{vmatrix} = -\begin{vmatrix} 2z^2 & y^2+z^2-x^2 \\ y^2+z^2-x^2 & 2y^2 \end{vmatrix}$$

$$= (y^2+z^2-x^2)^2 - 4y^2z^2$$
$$= (y^2+z^2-x^2+2yz)(y^2+z^2-x^2-2yz)$$
$$= \{(y+z)^2-x^2\}\{(y-z)^2-x^2\}$$
$$= -(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

10. Prove the identity

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} \equiv 2abc\,(a+b+c)^3.$$

Subtracting the last column from each of the others, $(a+b+c)^2$ may be taken out as a factor. Calling the remaining determinant Δ', and subtracting in it the sum of the first two rows from the last, we have

$$\Delta' = \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} = \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$= \frac{1}{ab}\begin{vmatrix} a(b+c-a) & 0 & a^2 \\ 0 & b(c+a-b) & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}.$$

Adding the last column to each of the others, we obtain

$$\Delta' = \frac{1}{ab}\begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(c+a) & b^2 \\ 0 & 0 & 2ab \end{vmatrix} = 2\begin{vmatrix} a(b+c) & a^2 \\ b^2 & b(c+a) \end{vmatrix} = 2ab\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$= 2abc(a+b+c).$$

Hence, $\Delta = \Delta'(a+b+c)^2 = 2abc(a+b+c)^3.$

11. Prove the identity

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \equiv (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma).$$

Subtracting the first column from each of the others, $\beta - \alpha$ and $\gamma - \alpha$ become factors. In the reduced determinant, subtract the first row multiplied by α^2 from the second row.

12. Resolve into simple factors the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \end{vmatrix}.$$

Proceeding as in Ex. 11, we easily find that $(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)$ is a factor, and that the reduced determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ \beta+\alpha & \gamma+\alpha & \delta+\alpha \\ \beta^3+\beta^2\alpha+\beta\alpha^2+\alpha^3 & \gamma^3+\gamma^2\alpha+\gamma\alpha^2+\alpha^3 & \delta^3+\delta^2\alpha+\delta\alpha^2+\alpha^3 \end{vmatrix}.$$

Subtracting the first column from each of the others, $(\gamma-\beta)(\delta-\beta)$ comes out as a factor, and the remaining factor is easily found to be $(\delta-\gamma)(\alpha+\beta+\gamma+\delta)$. Hence, finally,

$$\Delta = -(\beta-\gamma)(\alpha-\delta)(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)(\alpha+\beta+\gamma+\delta).$$

13. Resolve into linear factors the determinant

$$\Delta \equiv \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Multiply the second column by ω, and the third by ω^2; and add to the first. The factor $a+\omega b+\omega^2 c$ may then be taken off the first column (since $\omega^3=1$), leaving the constituents 1, ω, ω^2. Adding then the second and third rows to the first, the factor $a+b+c$ may be taken out; and the remaining determinant is easily found to be equal to $a+\omega^2 b+\omega c$. Hence we have

$$\Delta \equiv (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c).$$

14. Resolve into linear factors the determinant

$$\Delta \equiv \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

The result is as follows :—

$$\Delta = -(a+b+c+d)(b+c-a-d)(c+a-b-d)(a+b-c-d),$$

since each of the factors here written is a factor of the determinant ; for example, $a+b-c-d$ is shown to be a factor by adding the second column to the first, and subtracting the third and fourth. By comparing the sign of a^4 it appears that the negative sign must be attached to the product.

It may be observed that the determinant of Ex. 9 is a particular case of the determinant here considered, viz. that obtained by putting $a = 0$, as will appear by comparing the equivalent forms of Ex. 9, Art. 124.

131. **Multiplication of Determinants.**—Prop. VIII.—*The product of two determinants of any order is itself a determinant of the same order.*

We shall prove this for two determinants of the third order. The student will observe, from the nature of the proof, that it is equally applicable in general. We propose to show that the product of the two determinants $(a_1b_2c_3)$, $(\alpha_1\beta_2\gamma_3)$ is

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix};$$

whose constituents are the sums of the products of the constituents in any row of $(a_1b_2c_3)$ by the corresponding constituents in any row of $(\alpha_1\beta_2\gamma_3)$.

Since each column consists of the sum of three terms, this determinant can be expanded into the sum of 27 others (Art. 128). Now it will be observed that when any one of these is written down, a common factor can be taken off each column ; and that several of the partial determinants will, when these factors are removed, have two (or more) columns identical. The determinants which do not vanish in this way can be easily selected. Taking, for example, the first vertical line of the first

column; this would give a vanishing determinant if we were to take along with it the first line of the second column. We take then the second line of the second column, and along with these two we must take the third line of the third column to obtain a determinant which does not vanish. Retaining still the first line of the first column, we may take the third line of the second column along with the second line of the third column. Taking out the common factors of the columns, we write down these two determinants as follows:—

$$\alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

Taking in turn each of the other lines of the first column, we obtain four other determinants which do not vanish. Thus there are in all six terms; and it is plain that $(a_1b_2c_3)$ is a factor in each of these. Taking out this factor there remains the sum of six terms—

$$\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_3\beta_2\gamma_1,$$

and this is the determinant $(\alpha_1\beta_2\gamma_3)$. We have therefore proved that the determinant above written is the product of the two given determinants.

In either of the given determinants the rows may be written in place of columns; hence, the product may be written in several different forms as a determinant; but these will, of course, give the same value when expanded.

132. **Multiplication of Determinants continued.**—Another mode of proof of the proposition of the last Article, expressing as a determinant the product of two given determinants of the same order, may be derived from Laplace's mode of development already explained (Art. 125).

The nature of this proof will be sufficiently understood from the application which follows to two determinants of the third order:—

The product of the two determinants $(a_1 b_2 c_3)$, $(\alpha_1 \beta_2 \gamma_3)$ is (see Ex. 3, Art. 125) plainly equal to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & -1 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & -1 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}.$$

In this determinant add to the fourth column the sum of the first multiplied by α_1, the second by β_1, and the third by γ_1; add to the fifth column the sum of the first multiplied by α_2, the second by β_2, and the third by γ_2; and add to the sixth column the sum of the first multiplied by α_3, the second by β_3, and the third by γ_3. The determinant becomes then

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2 & b_2 & c_2 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3 & b_3 & c_3 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}.$$

And this is, by Art. 125, equal to the product (with the proper sign) of the determinant

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \text{(which is equal to } -1)$$

by the complementary minor, which is the same determinant as that obtained in the preceding Article. That the sign to be attached to the product is negative is easily seen by moving down the first three rows till the diagonals of the two minors in question form the diagonal of the determinant itself. The student will have no difficulty in observing that, in the general case, the number of such displacements is odd when the order of the given determinants is odd, and even when it is even; so that the sign to be placed before the product-determinant of Art. 131 is always positive.

Examples.

1. Show that the product of the two determinants

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}, \quad \begin{vmatrix} a'-ib' & c'-id' \\ -c'-id' & a'+ib' \end{vmatrix},$$

where $i = \sqrt{-1}$, may be written in the form

$$\begin{vmatrix} D-iC & B-iA \\ -B-iA & D+iC \end{vmatrix};$$

where

$$A \equiv bc' - b'c + ad' - a'd, \qquad B \equiv ca' - c'a + bd' - b'd,$$

$$C \equiv ab' - a'b + cd' - c'd, \qquad D \equiv aa' + bb' + cc' + dd';$$

and hence prove Euler's theorem

$$\begin{aligned} &(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2) \\ &\equiv (aa'+bb'+cc'+dd')^2 + (bc'-b'c+ad'-a'd)^2 \\ &+ (ca'-c'a+bd'-b'd)^2 + (ab'-a'b+cd'-c'd)^2, \end{aligned}$$

viz. *the product of two sums each of four squares can be expressed as the sum of four squares.*

2. Prove the following expression for the square of a determinant of the third order :—

$$2\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}^2 = \begin{vmatrix} 2(ac-b^2) & ac'+a'c-2bb' & ac''+a''c-2bb'' \\ ac'+a'c-2bb' & 2(a'c'-b'^2) & a'c''+a''c'-2b'b'' \\ ac''+a''c-2bb'' & a'c''+a''c'-2b'b'' & 2(a''c''-b''^2) \end{vmatrix}.$$

This appears by multiplying the two determinants

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}, \quad \begin{vmatrix} c & -2b & a \\ c' & -2b' & a' \\ c'' & -2b'' & a'' \end{vmatrix},$$

which differ only by the factor 2.

3. Prove the identity

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \equiv (a^3+b^3+c^3-3abc)^2.$$

This may be readily proved by multiplying together the two equivalent determinants

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}, \quad \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}.$$

4. Show that two determinants of different orders may be multiplied together.

For their orders may be made equal; since the order of any determinant can be increased by adding any number of columns and the same number of rows consisting of units in the diagonal, and all the rest zero constituents. For example,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \text{may be written} \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{vmatrix},$$

the only effect of the added constituents being to multiply the determinant by unity. More generally, one set of added constituents (*i. e.* those either to the right or the left of the diagonal) might be taken to be any quantities whatever, the remaining set being ciphers. Thus $(a_1 b_2)$ may be written in either of the forms

$$\begin{vmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \delta & \epsilon \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & \delta & a_1 & b_1 \\ 0 & \epsilon & a_2 & b_2 \end{vmatrix};$$

as readily appears by means of the expansion of Art. 124.

133. **Rectangular Arrays.**—Arrays in which the number of rows is not equal to the number of columns may be called *rectangular*. These do not themselves represent any definite function; but if two such arrays of the same dimensions are given, there can be derived from them by the process of Art. 131 a determinant whose value we proceed to investigate.

(1). *When the number of columns exceeds the number of rows.*

Take, for example, the two rectangular arrays,

$$\left.\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{matrix}\right\} (2);$$

and performing on these a process similar to that employed in multiplying two determinants, we obtain the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 + d_1\delta_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 + d_1\delta_2 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 + d_2\delta_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 + d_2\delta_2 \end{vmatrix}.$$

The value of this is easily found to be

$$(a_1b_2)(\alpha_1\beta_2) + (a_1c_2)(\alpha_1\gamma_2) + (a_1d_2)(\alpha_1\delta_2) + (b_1c_2)(\beta_1\gamma_2) \\ + (b_1d_2)(\beta_1\delta_2) + (c_1d_2)(\gamma_1\delta_2),$$

i.e. *the sum of the products of all possible determinants which can be formed from one array (by taking a number of columns equal to the number of rows) multiplied by the corresponding determinants formed from the other array.*

Another proof of this proposition, analogous to the treatment of multiplication of determinants in Art. 132, is given in the sixth of the following examples; and either of these proofs can be easily generalised.

(2). *When the number of rows exceeds the number of columns the resulting determinant vanishes.*

Take, for example, the two arrays,

$$\left.\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix}\right\} (2).$$

Performing the process of multiplication, we have

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}.$$

It will be observed that this determinant is the same as would arise if a column of ciphers were added to each of the given arrays, and the determinants so formed then multiplied. It follows that the determinant vanishes.

A similar proof applies in general. It is only necessary in any instance to add to each array columns of ciphers, so as to make the number of columns equal to the number of rows, and then multiply the two determinants.

Examples.

1. From the two arrays

$$\left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\} (1), \qquad \left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\} (2),$$

prove

$$\begin{vmatrix} 3 & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} \equiv (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2.$$

2. From the two arrays

$$\left.\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right\} (1), \qquad \left.\begin{matrix} c & -2b & a \\ c' & -2b' & a' \end{matrix}\right\} (2),$$

prove

$$4(ac-b^2)(a'c'-b'^2) - (ac'+a'c-2bb')^2 \equiv 4(bc'-b'c)(ab'-a'b) - (ac'-a'c)^2$$

3. By squaring the array

$$\left.\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right\},$$

prove

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2) = (aa'+bb'+cc')^2 + (bc'-b'c)^2 + (ca'-c'a)^2 + (ab'-a'b)^2.$$

4. Verify, by squaring the array

$$\left.\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \end{matrix}\right\},$$

the result of Ex. 1, Art. 132.

5. Prove the determinant identity

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix} \equiv 0.$$

This can be proved by multiplying the two arrays

$$\left.\begin{matrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \\ a_4^2 & a_4 & 1 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \\ 1 & -2b_4 & b_4^2 \end{matrix}\right\} (2).$$

6. Find the value of the following determinant, and hence derive another proof of the property of arrays of the first kind—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ -1 & 0 & 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & -1 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & -1 & 0 & \gamma_1 & \gamma_2 \\ 0 & 0 & 0 & -1 & \delta_1 & \delta_2 \end{vmatrix}.$$

Expanding this by Laplace's method, we readily find its value to be the six products, $\Sigma(a_1b_2)(\alpha_1\beta_2)$, of p. 278; and treating the determinant as in Art. 132, viz. adding to the fifth column the sum of the first multiplied by α_1, the second by β_1, &c., we reduce it to the determinant of the second order at the top of p. 278.

134. **Solution of a System of Linear Equations.**—We have seen in Art. 124 that a determinant may be expanded as a linear homogeneous function of the constituents in any row or column, the coefficient of any constituent being the corresponding minor with its proper sign. We have, for example,

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + \&c.$$

Now, the coefficients A_1, A_2, &c., are connected with the constituents of the other columns by $n - 1$ identical relations, viz.

$$b_1A_1 + b_2A_2 + b_3A_3 + \&c. \equiv 0.$$

$$c_1A_1 + c_2A_2 + c_3A_3 + \&c. \equiv 0, \&c.;$$

for any one of these is what the determinant becomes when the constituents of the corresponding column are substituted for a_1, a_2, a_3, &c., and must therefore vanish.

By the aid of these relations we can write down the solution of a system of linear equations. The following application to the case of three unknown quantities x, y, z, is sufficient to explain the general process. Let the equations be

$$a_1x + b_1y + c_1z = m_1,$$

$$a_2x + b_2y + c_2z = m_2,$$

$$a_3x + b_3y + c_3z = m_3.$$

Multiply the first equation by A_1, the second by A_2, and the third by A_3; and add. The coefficients of y and z vanish, in virtue of the relations above proved, and we obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x = m_1A_1 + m_2A_2 + m_3A_3,$$

or

$$\Delta x = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix},$$

where Δ represents the determinant formed from the nine constituents a_1, b_1, c_1, &c.

Similarly, multiplying by B_1, B_2, B_3, we obtain

$$(b_1B_1 + b_2B_2 + b_3B_3)\,y = m_1B_1 + m_2B_2 + m_3B_3,$$

$$\Delta y = \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix},$$

where the determinant on the right-hand side is what Δ becomes when m_1, m_2, m_3 are substituted for the constituents of the second column. Similarly, we obtain for z

$$\Delta z = \begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix}.$$

These values may be written more compactly, as follows:—

$$\Delta x = (m_1b_2c_3), \quad \Delta y = (a_1m_2c_3), \quad \Delta z = (a_1b_2m_3).$$

In general, the values of x, y, z, &c., may be written as follows:—

$$x = \frac{(m_1\,b_2\,c_3\,.\,.\,.\,l_n)}{(a_1\,b_2\,c_3\,.\,.\,.\,l_n)}, \quad y = \frac{(a_1\,m_2\,b_3\,.\,.\,.\,l_n)}{(a_1\,b_2\,c_3\,.\,.\,.\,l_n)}, \quad z = \frac{(a_1\,b_2\,m_3\,.\,.\,.\,l_n)}{(a_1\,b_2\,c_3\,.\,.\,.\,l_n)}, \text{ \&c.}$$

where, to obtain the value of any unknown, the known quantities m_1, m_2, &c., on the right-hand side of the given equations are to be substituted in Δ for the coefficients of the required unknown, and the determinant so formed to be divided by Δ.

135. **Linear Homogeneous Equations.**—When $n-1$ linear homogeneous equations between n variables are given, the ratios of the variables can be determined by bringing any one of them to the right-hand side of the equations, and solving as in the previous Article; or we may determine these ratios more conveniently, as follows. We take the particular case of three equations between four quantities x, y, z, w, which will be sufficient to illustrate the general process:

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1w = 0 \\ a_2x + b_2y + c_2z + d_2w = 0 \\ a_3x + b_3y + c_3z + d_3w = 0 \end{aligned}\right\}. \qquad (1)$$

To these may be added a fourth equation whose coefficients are undetermined, viz.

$$a_4x + b_4y + c_4z + d_4w = \lambda. \qquad (2)$$

Calling $(a_1b_2c_3d_4)$ as usual Δ, and solving from these four equations by the method of the last Article, we obtain, since $m_1 = 0$, $m_2 = 0$, $m_3 = 0$, $m_4 = \lambda$, the following values:—

$$\Delta x = \lambda A_4, \quad \Delta y = \lambda B_4, \quad \Delta z = \lambda C_4, \quad \Delta w = \lambda D_4,$$

or,

$$\frac{x}{A_4} = \frac{y}{B_4} = \frac{z}{C_4} = \frac{w}{D_4} = \frac{\lambda}{\Delta}. \qquad (3)$$

The first three of these equations express the ratios of x, y, z, w in terms of the coefficients in the three given equations. And, in general, *the variables are proportional to the coefficients in the expansion of Δ of the constituents of the n^{th} row supposed added to the $n-1$ rows resulting from the given equations.*

We can now express the condition that n linear homogeneous equations should be consistent with one another; for example, that the equation (2) should, when $\lambda = 0$, be consistent with the equations (1). We have only to substitute in (2) the ratios derived from (1), when we obtain

$$a_4A_4 + b_4B_4 + c_4C_4 + d_4D_4 = 0,$$

or

$$\Delta = 0.$$

The same thing appears from the equations (3); for if $\lambda = 0$, and if x, y, z, w do not all vanish, Δ must vanish.

What has been proved may be expressed as follows:—*The result of eliminating n quantities between n equations linear and homogeneous in these quantities is the vanishing of the determinant formed by the coefficients of the given equations.*

136. **Reciprocal Determinants.** — The quantities $A_1, B_1, C_1 \ldots A_2, B_2$, &c. (Art. 124), which occur in the expansion of a determinant (*i. e.* the first minors with their proper signs), may be called *inverse constituents*; and the determinant formed with them the inverse or *reciprocal determinant*. We proceed to prove certain useful relations connecting the two determinants.

(1). *To express the reciprocal in terms of the given determinant.* Let the reciprocal of Δ be denoted by Δ', and multiply the two determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

All the constituents of the resulting determinant except those in the diagonal vanish (Art. 134); and the result is

$$\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3;$$

whence

$$\Delta' = \Delta^2.$$

The process here employed in the particular case of two determinants of the third order is equally applicable in general; giving $\Delta\Delta' = \Delta^n$, or $\Delta' = \Delta^{n-1}$. Hence *the reciprocal determinant is equal to the $(n-1)^{th}$ power of the given determinant.*

(2) *To express any minor of the reciprocal determinant in terms of the original constituents.*

We take, for example, the determinant of the fourth order and proceed to express the first minors of its reciprocal. Multiplying the two determinants on the left-hand side of the following equation, and employing the identical equations of Art. 134, we obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & \Delta & 0 & 0 \\ a_3 & 0 & \Delta & 0 \\ a_4 & 0 & 0 & \Delta \end{vmatrix};$$

whence

$$\Delta \begin{vmatrix} B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix} = a_1\Delta^3,$$

or

$$(B_2C_3D_4) = a_1\Delta^2,$$

thus expressing the first minor of Δ' complementary to A_1.

Again, to express the second minors of Δ', we have, by an exactly similar process,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & \Delta & 0 \\ a_4 & b_4 & 0 & \Delta \end{vmatrix};$$

whence

$$\Delta \begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \Delta^2,$$

or

$$(C_3D_4) = (a_1b_2)\,\Delta.$$

The general theorem may be expressed as follows:—*A minor of the order m formed out of the inverse constituents is equal to the complementary of the corresponding minor of the original determinant* Δ *multiplied by the* $(m-1)^{th}$ *power of* Δ.

The method of proof above given can be generalized. In the case of a determinant of the fifth order, for example, the student will easily verify the following expression for a minor of the third order:—

$$(C_3D_4E_5) = (a_1b_2)\,\Delta^2.$$

If the original determinant Δ vanishes, it is plain that not only the reciprocal determinant itself, but also all its minors of any order vanish. The vanishing of the minors of the second order may be expressed in the following useful form:—*When a determinant vanishes, the constituents of any row of its reciprocal are proportional to those of any other row, and the constituents of any column to those of any other column.*

137. **Symmetrical Determinants.**—Two constituents of a determinant are said to be *conjugate* when one occupies with reference to the leading constituent the same position in the rows as the other does in the columns. For example, d_2 and b_4 are conjugates, one occupying the fourth place in the second row, and the other the fourth place in the second column. Each of the leading constituents is its own conjugate. Any two conjugate constituents are situated in a line perpendicular to the principal diagonal, and at equal distances from it on opposite sides.

A *symmetrical* determinant is one in which every two conjugate constituents are equal to each other. For examples of such determinants the student may refer to Art. 124, Exs. 2, 9, 10, and Art. 125, Ex. 4.

In a symmetrical determinant the first minors complementary to any two conjugate constituents are equal, since they differ only by an interchange of rows and columns. The corresponding inverse constituents are also equal, the signs to be attached to the minors being the same in both cases. It follows that the *reciprocal of a symmetrical determinant is itself symmetrical.*

The leading minors are all symmetrical determinants.

The mode of expansion of Art. 127 is especially useful in the case of symmetrical determinants, as will appear from the examples which follow.

EXAMPLES.

1. Form the reciprocal of the symmetrical determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Using the capital letters to denote the reciprocal constituents as explained in Art. 124, so that Δ may be expanded in any one of the forms $aA + hH + gG$, $hH + bB + fF$, $gG + fF + cC$, we may write the reciprocal determinant Δ' as follows :—

$$\Delta' \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \equiv \begin{vmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ca - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{vmatrix}.$$

2. Form similarly the reciprocal of

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}.$$

Using a notation similar to that of the preceding example, so that Δ may be expanded indifferently in any of the forms

$$aA + hH + gG + lL, \quad hH + bB + fF + mM, \text{ \&c.},$$

the reciprocal determinant Δ' is obtained by replacing in Δ the constituents by the corresponding capital letters. The student will find no difficulty in writing out, if necessary, the expanded form of any of the reciprocal constituents; for example, F is the minor complementary to f with its proper sign (the negative sign in this case), and F is therefore obtained from the expansion of

$$-\begin{vmatrix} a & h & l \\ g & f & n \\ l & m & d \end{vmatrix}.$$

3. Expand the determinant Δ of Ex. 10, Art. 124, by the method of Art. 127, Bringing the last row and last column into the positions of first row and first

column, and using the notation of Ex. 1 for the inverse constituents of the leading minor, the result can be written down at once in the form

$$-\Delta = A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu.$$

Since a determinant is unaltered when both rows and columns are written in reverse order, if the expansion of a determinant be required in terms of the last row and last column (as in the present example), it is not necessary to move them in the first instance into the positions of first row and first column. The expansion can be written down from the determinant as it stands, replacing in the rule of Art. 127 the leading constituent and its minor by the last diagonal constituent and its complementary minor.

4. Expand the determinant Δ of the above Ex. 2, in terms of the last row and column, by the method of Art. 127.

Attending to the remark at the end of the preceding example, and using A, B, C, F, G, H, to represent the same quantities as in Exs. 1 and 3, the result may be written down as follows:—

$$\Delta = d\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} - Al^2 - Bm^2 - Cn^2 - 2Fmn - 2Gnl - 2Hlm.$$

When a symmetrical determinant of any order is bordered symmetrically (*i.e.* by the same constituents horizontally and vertically) the result is plainly a symmetrical determinant of the next higher order. The result of Art. 127 shows in general that the expansion of the bordered determinant consists of the original determinant multiplied by the constituent common to the added row and column, together with a homogeneous function of the second degree of the remaining added constituents.

5. Expand the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g & l & \alpha \\ h & b & f & m & \beta \\ g & f & c & n & \gamma \\ l & m & n & d & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}.$$

This is the determinant of Ex. 2, bordered symmetrically, the common constituent of the added lines being zero. The result is plainly a homogeneous function of the second degree of α, β, γ, δ; and, by aid of the notation of Ex. 2, the value of $-\Delta$ may be written down at once in the form

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta + 2L\alpha\delta + 2M\beta\delta + 2N\gamma\delta.$$

6. Prove by means of the Proposition of Art. 131, that the square of any determinant is a symmetrical determinant.

7. The product of two reciprocal determinants is the reciprocal determinant of the product of the two original determinants.

138. **Skew-Symmetric and Skew Determinants.**—A *skew-symmetric* determinant is one in which every constituent is equal to its conjugate with sign changed. Since each leading constituent is its own conjugate, it follows that in such a determinant all the leading diagonal constituents are zero.

A determinant in which all except the leading constituents are equal to their conjugates with sign changed is called a *skew* determinant. Thus, while a skew-symmetric determinant is zero-axial, a skew determinant is not. The calculation of a skew determinant may plainly be reduced to that of skew-symmetric determinants by the method of Art. 126.

The remainder of this Article will be occupied with the proof of certain useful properties of skew-symmetric determinants.

(1). *A skew-symmetric determinant of odd order vanishes.*

For any skew-symmetric determinant Δ is unaltered by changing the columns into rows, and then changing the signs of all the rows. But when the order of the determinant is odd, this process ought to change the sign of Δ; hence Δ must in this case vanish. For example,

$$\Delta \equiv \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} \equiv 0.$$

(2). *The reciprocal of a skew-symmetric determinant of the n^{th} order is a symmetric determinant when n is odd, and a skew-symmetric determinant when n is even.*

In any skew-symmetric determinant the minors corresponding to a pair of conjugate constituents differ by an interchange of rows and columns, and by the signs of all the constituents. Hence the two minors are equal when their order is even, namely when n is odd; and equal with opposite signs when n is even. In the former case, therefore, the reciprocal determinant is symmetric; and in the latter case it is skew-symmetric, its leading diagonal constituents all being skew-symmetric determinants of odd order.

(3). *A skew-symmetric determinant of even order is a perfect square.*

This follows from the principles established in Art. 136.

Take, for example, the determinant of the fourth order

$$\Delta = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix};$$

and let the inverse constituents forming its reciprocal be denoted by A_1, B_1, ... A_2, &c. We have then, by (2), Art. 136,

$$A_1 B_2 - A_2 B_1 = \Delta \begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} = f^2 \Delta.$$

Now A_1 and B_2, being skew-symmetric determinants of odd order, vanish; and $A_2 = -B_1$, since these are conjugate minors; hence $f^2 \Delta = A_2^2$, which proves that Δ is a perfect square. Similarly, for the determinant of the sixth order, it is proved that the product of Δ by a skew-symmetric determinant of the fourth order is a perfect square; and since the latter determinant has been just proved to be a perfect square, it follows that Δ is also. By an exactly similar process, assuming the truth of the Proposition for the determinant of the sixth order, it follows for one of the eighth; and so on.

Examples.

1. Verify the following expression for the skew-symmetric determinant of the fourth order:—

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} \equiv (af - be + cd)^2$$

2. Expand in powers of x the skew determinant

$$\Delta \equiv \begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix}$$

When the expansion of Art. 126 is employed to calculate a skew determinant, it is to be observed that the determinants of odd order in the expansion all vanish, and those of even order may be expressed as squares. Here the coefficients of the odd powers of x plainly vanish, and the result takes the form

$$\Delta \equiv x^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)\,x^2 + (af - be + cd)^2.$$

3. Expand the skew determinant

$$\begin{vmatrix} A & a & b & c & d \\ -a & B & e & f & g \\ -b & -e & C & h & i \\ -c & -f & -h & D & j \\ -d & -g & -i & -j & E \end{vmatrix}.$$

The result may be written in the form

$$ABCDE + \Sigma j^2 ABC + \Sigma\,(ej - fi + gh)^2 A,$$

where the first Σ includes ten terms similar to the one here written, and the second Σ includes five terms. The terms involving the products in pairs of the leading constituents vanish, as also the term not involving these quantities.

4. The square of any determinant of even order can be expressed as a skew-symmetric determinant.

The following method of proof is applicable in general.

The square of $(a_1 b_2 c_3 d_4)$ is obtained by multiplying the two following determinants:—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} -b_1 & a_1 & -d_1 & c_1 \\ -b_2 & a_2 & -d_2 & c_2 \\ -b_3 & a_3 & -d_3 & c_3 \\ -b_4 & a_4 & -d_4 & c_4 \end{vmatrix};$$

and the product of these is

$$\begin{vmatrix} 0, & -(a_1b_2)-(c_1d_2), & -(a_1b_3)-(c_1d_3), & -(a_1b_4)-(c_1d_4), \\ (a_1b_2)+(c_1d_2), & 0, & -(a_2b_3)-(c_2d_3), & -(a_2b_4)-(c_2d_4), \\ (a_1b_3)+(c_1d_3), & (a_2b_3)+(c_2d_3), & 0, & -(a_3b_4)-(c_3d_4), \\ (a_1b_4)+(c_1d_4), & (a_2b_4)+(c_2d_4), & (a_3b_4)+(c_3d_4), & 0, \end{vmatrix}.$$

which is a skew-symmetric determinant.

5. Form the reciprocal of a skew-symmetric determinant of the third order.

Using for Δ the form in (1) of the present Article, the result is easily found to be the symmetric determinant

$$\begin{vmatrix} c^2 & -bc & ac \\ -bc & b^2 & -ab \\ ac & -ab & a^2 \end{vmatrix}.$$

6. Form the reciprocal of the skew-symmetric determinant Δ of the fourth order in Ex. 1.

Representing by ϕ the function $af - be + cd$ whose square is equal to Δ, and by Δ' the required reciprocal, we easily find

$$\Delta' = \begin{vmatrix} 0 & f\phi & -e\phi & d\phi \\ -f\phi & 0 & c\phi & -b\phi \\ e\phi & -c\phi & 0 & a\phi \\ -d\phi & b\phi & -a\phi & 0 \end{vmatrix}.$$

The value of this skew-symmetric determinant may be written down by aid of the result of Ex. 1. It is thus immediately verified that $\Delta' = (af - be + cd)^2\phi^4 = \Delta^3$.

7. Form the reciprocal of the skew-symmetric determinant Δ of the fifth order, obtained by making the leading coefficients all vanish in the determinant of Ex. 3.

Since the reciprocal is a symmetric determinant (see (2), Art. 138), and since also it must be such that the constituents of any line are proportional to those of any parallel line (Art. 136), it appears that the required determinant must be of the form

$$\begin{vmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & \phi_1\phi_4 & \phi_1\phi_5 \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & \phi_2\phi_4 & \phi_2\phi_5 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3^2 & \phi_3\phi_4 & \phi_3\phi_5 \\ \phi_4\phi_1 & \phi_4\phi_2 & \phi_4\phi_3 & \phi_4^2 & \phi_4\phi_5 \\ \phi_5\phi_1 & \phi_5\phi_2 & \phi_5\phi_3 & \phi_5\phi_4 & \phi_5^2 \end{vmatrix},$$

in which $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are five functions of the second degree in the original constituents whose squares are the values of the five first minors complementary to the leading constituents of Δ.

In general the reciprocal of a skew-symmetric determinant of any odd order $2m+1$ is of a form similar to that just written, the diagonal constituents being the squares, and the remaining constituents the products in pairs, of $2m+1$ functions, each of the m^{th} degree in the original constituents.

139. **Theorem.**—We conclude the present chapter with an important theorem relating to a determinant whose leading first minor vanishes. Adopting the notation of Art. 127, we regard Δ as the vanishing determinant, and state the theorem to be proved as follows:—*If a determinant Δ, whose value is zero, be bordered in any manner, the product of the determinant so formed by the leading first minor of Δ is equal to the product of two linear homogeneous functions of the added constituents.*

Retaining the notation of Art. 127, we shall prove that the product of Δ' and A_1 may be expressed in the form:—

$$A_1\Delta' = -(A_1\alpha + B_1\beta + C_1\gamma + \ldots)(A_1\alpha' + A_2\beta' + A_3\gamma' + \ldots).$$

This follows at once from (2) of Art. 136 by considering in the determinant reciprocal to Δ' the values of the constituents inverse to a_0 α, α', a_1; and expressing in terms of the original constituents the determinant of the second order formed by these four. Another proof of this result may be readily derived from the expansion of Art. 127, by the aid of the property of the reciprocal of a vanishing determinant (Art. 136), viz. that in the determinant formed by A_1, B_1, C_1, &c., the constituents in any line are proportional to those in any parallel line.

If the determinant Δ is symmetrical, and the bordering also symmetrical, the two factors on the right-hand side of the above equation become identical, and the theorem takes the following form:—*If a symmetrical determinant, whose value is zero, be bordered symmetrically, the product of the determinant so formed by its leading second minor is equal to the square with negative sign of a linear homogeneous function of the bordering constituents.*

Regarding Δ' as the original determinant, the following useful statement may be given to the theorem just proved:—*If in any symmetrical determinant the leading first minor vanish, the determinant itself and its leading second minor have opposite signs.*

Examples.

1. If a skew-symmetric determinant Δ of odd order $2m+1$ be bordered in any manner, the resulting determinant Δ' is equal to the product of two rational functions each containing the added constituents in the first degree, and the original constituents in the m^{th} degree.

Writing, according to the result of Ex. 7, Art. 138, the reciprocal of the given skew-symmetric determinant in the form

$$\begin{vmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & . \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & . \\ . & . & . & . \end{vmatrix},$$

and applying the theorem of the present Article, we find

$$\phi_1^2\Delta' = -(\phi_1^2\alpha + \phi_1\phi_2\beta + \phi_1\phi_3\gamma + \ldots)(\phi_1^2\alpha' + \phi_2\phi_1\beta' + \phi_3\phi_1\gamma' + \ldots),$$

or $$\Delta' = -(\phi_1\alpha + \phi_2\beta + \phi_3\gamma + \ldots)(\phi_1\alpha' + \phi_2\beta' + \phi_3\gamma' + \ldots).$$

It may be observed that if in this result α', β', γ', &c., be made equal to $-\alpha$, $-\beta$, $-\gamma$, &c., respectively, we fall back on the theorem (3) of Art. 138.

2. If a skew-symmetric determinant of even order $2m$ be bordered in any manner, the resulting determinant is equal to the product of two rational functions, one of the m^{th}, and the other of the $(m+1)^{th}$ degree in the constituents.

This may be derived immediately from the last example by making therein all the added constituents α', β', γ', &c., equal to zero, except the last, which is to be made $= 1$. The determinant then reduces to one of the $(2m+1)^{th}$ order of the kind here considered, the bordering constituents forming the top row and the last column. It appears also that the factor of the m^{th} degree in the result is the square root of the given skew-symmetric determinant of order $2m$.

3. Prove

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ \alpha' & 0 & c & -b \\ \beta' & -c & 0 & a \\ \gamma' & b & -a & 0 \end{vmatrix} = -(a\alpha + b\beta + c\gamma)(a\alpha' + b\beta' + c\gamma').$$

4. Resolve into its factors

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma & \delta \\ \alpha' & 0 & c & -b & x \\ \beta' & -c & 0 & a & y \\ \gamma' & b & -a & 0 & z \\ \delta' & -x & -y & -z & 0 \end{vmatrix}.$$

Ans. $(ax + by + cz)\{x(\beta\gamma') + y(\gamma\alpha') + z(\alpha\beta') + a(\alpha\delta') + b(\beta\delta') + c(\gamma\delta')\}$.

Miscellaneous Examples.

1. Prove

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \equiv J,$$

where J has the same signification as in Art. 37.

2. Prove

$$\begin{vmatrix} \beta+\gamma & \gamma+\alpha & \alpha+\beta \\ \beta'+\gamma' & \gamma'+\alpha' & \alpha'+\beta' \\ \beta''+\gamma'' & \gamma''+\alpha'' & \alpha''+\beta'' \end{vmatrix} \equiv 2 \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}.$$

3. Prove

$$\begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix} \equiv (\beta\gamma')\,(\gamma\alpha')\,(\alpha\beta'),$$

where the factors on the right-hand side are determinants of the second order.

Dividing the rows by $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$; and putting $\lambda = \frac{\alpha}{\alpha'}$, $\mu = \frac{\beta}{\beta'}$, $\nu = \frac{\gamma}{\gamma'}$, the determinant (omitting a factor) reduces to the form

$$\begin{vmatrix} 1 & \mu+\nu & \mu\nu \\ 1 & \nu+\lambda & \nu\lambda \\ 1 & \lambda+\mu & \lambda\mu \end{vmatrix} \equiv \begin{vmatrix} 1 & -\lambda & \mu\nu \\ 1 & -\mu & \nu\lambda \\ 1 & -\nu & \lambda\mu \end{vmatrix} \equiv -(\mu-\nu)(\nu-\lambda)(\lambda-\mu), \text{ \&c.}$$

4. Find the value of the determinant

$$\begin{vmatrix} 1 & \beta+\gamma+\delta & \beta\gamma+\beta\delta+\gamma\delta & \beta\gamma\delta \\ 1 & \alpha+\gamma+\delta & \alpha\gamma+\alpha\delta+\gamma\delta & \alpha\gamma\delta \\ 1 & \alpha+\beta+\delta & \alpha\beta+\alpha\delta+\beta\delta & \alpha\beta\delta \\ 1 & \alpha+\beta+\gamma & \alpha\beta+\alpha\gamma+\beta\gamma & \alpha\beta\gamma \end{vmatrix}.$$

Since the interchange of two letters would make two rows identical, this can differ by a numerical factor only from the product of the six differences. Or we may reduce the determinant easily to the form in Ex. 10, Art. 122. The value of a similar determinant of any order can be found in the same way; and the sign can be determined in any instance by the method of Ex. 9, Art. 122.

5. Prove

$$\begin{vmatrix} \beta^2\gamma^2+\alpha^2\delta^2 & \beta\gamma+\alpha\delta & 1 \\ \gamma^2\alpha^2+\beta^2\delta^2 & \gamma\alpha+\beta\delta & 1 \\ \alpha^2\beta^2+\gamma^2\delta^2 & \alpha\beta+\gamma\delta & 1 \end{vmatrix} = (\beta-\gamma)(\alpha-\delta)(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta).$$

Add the last column multiplied by $2\alpha\beta\gamma\delta$ to the first. The determinant becomes then of the form of Ex. 9, Art. 122.

6. Prove

$$\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix} \equiv 64(\beta-\gamma)(\alpha-\delta)(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta).$$

7. Prove

$$\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix} \equiv -(ac-b^2)(ax^2+2bx+c).$$

Subtract from the third row the second row plus the first multiplied by x.

8. Prove similarly

$$\begin{vmatrix} a & b & c & ax^2+2bx+c \\ b & c & d & bx^2+2cx+d \\ c & d & e & cx^2+2dx+e \\ ax^2+2bx+c & bx^2+2cx+d & cx^2+2dx+e & 0 \end{vmatrix}$$

$$\equiv - \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} (ax^4+4bx^3+6cx^2+4dx+e).$$

9. Given

$$f_1(x) = a_1x^3 + 3b_1x^2 + 3c_1x + d_1,$$

$$f_2(x) = a_2x^3 + 3b_2x^2 + 3c_2x + d_2,$$

$$f_3(x) = a_3x^3 + 3b_3x^2 + 3c_3x + d_3;$$

prove the identity

$$\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix} \equiv -18 \begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

The first determinant reduces easily (omitting a factor) to the following:—

$$\begin{vmatrix} a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix}.$$

We have seen (Ex. 4, Art. 132) that the order of a determinant may be increased without altering its value. By a suitable selection of the added constituents the calculation of a determinant may often be simplified by bordering it in this way. The determinant last written is plainly equal to

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2 & a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3 & a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix}.$$

Subtracting from the second column the first multiplied by x; subtracting then from the third the new second column multiplied by x; and, finally, from the fourth the new third column multiplied by x, we have the result above stated.

10. Show that the determinant

$$\begin{vmatrix} \lambda x^2 + cy^2 + bz^2 - 1 & (\lambda - c)\,xy & (\lambda - b)\,xz \\ (\lambda - c)\,xy & \lambda y^2 + az^2 + cx^2 - 1 & (\lambda - a)\,yz \\ (\lambda - b)\,xz & (\lambda - a)\,yz & \lambda z^2 + bx^2 + cy^2 - 1 \end{vmatrix}$$

contains $\lambda\,(x^2 + y^2 + z^2) - 1$ as a factor, and that the remaining factor is independent of λ.

Border the determinant, as in Ex. 9, with a first column whose constituents are $1, \lambda x, \lambda y, \lambda z$; and with a first row whose constituents are $1, 0, 0, 0$. Subtract then x times the first column from the second, y times the first column from the third, and z times the first column from the fourth. In the determinant thus altered subtract from the first row x times the second, plus y times the third, plus z times the fourth; and the result follows.

11. Expand in powers of x the determinant

$$\begin{vmatrix} a_1 + x & b_1 & c_1 & d_1 \\ a_2 & b_2 + x & c_2 & d_2 \\ a_3 & b_3 & c_3 + x & d_3 \\ a_4 & b_4 & c_4 & d_4 + x \end{vmatrix}.$$

Ans. $x^4 + (a_1 + b_2 + c_3 + d_4)\,x^3 + \{(b_2\,c_3) + (a_1\,d_4) + (a_1\,c_3) + (b_2d_4) + (a_1b_2) + (c_3d_4)\}\,x^2$
$+ \{(b_2c_3d_4) + (a_1c_3d_4) + (a_1b_2d_4) + (a_1b_2c_3)\}\,x + (a_1b_2c_3d_4)$.

12. Prove

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \dfrac{a^2}{a'} & \dfrac{b^2}{b'} & \dfrac{c^2}{c'} & \dfrac{d^2}{d'} \\ \dfrac{a'^2}{a} & \dfrac{b'^2}{b} & \dfrac{c'^2}{c} & \dfrac{d'^2}{d} \end{vmatrix} \equiv \frac{-(bc')(ad')(ca')(bd')(ab')(cd')}{abcd\,a'b'c'd'}.$$

13. Prove the identities

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} \equiv \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} \equiv \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} \equiv \begin{vmatrix} A & B \\ A' & B' \end{vmatrix},$$

where

$$A = (\beta-\gamma)(\alpha-\delta), \quad B = (\gamma-\alpha)(\beta-\delta), \quad C = (\alpha-\beta)(\gamma-\delta),$$
$$A' = (\beta'-\gamma')(\alpha'-\delta'), \quad B' = (\gamma'-\alpha')(\beta'-\delta'), \quad C' = (\alpha'-\beta')(\gamma'-\delta').$$

Expanding the first determinant in terms of the minors formed from the first two columns (see Art. 125), we easily prove that it is equal to

$$A(\beta'\gamma' + \alpha'\delta') + B(\gamma'\alpha' + \beta'\delta') + C(\alpha'\beta' + \gamma'\delta');$$

and employing the identical equation $A + B + C = 0$, along with the relations of Ex. 18, Art. 27, the result follows.

14. Prove that the determinant of Ex. 13 is equal to

$$\begin{vmatrix} 1 & \beta\gamma + \alpha\delta & \beta'\gamma' + \alpha'\delta' \\ 1 & \gamma\alpha + \beta\delta & \gamma'\alpha' + \beta'\delta' \\ 1 & \alpha\beta + \gamma\delta & \alpha'\beta' + \gamma'\delta' \end{vmatrix}.$$

This follows at once from the relations of Ex. 18, Art. 27. If α', β', γ', δ' be put equal to α^m, β^m, γ^m, δ^m in the result, we obtain an identity which includes Ex. 5, p. 295, as a particular case.

15. Express as a function of differences the following determinant, whose vanishing expresses the condition for involution of six points on a line:—

$$\Delta \equiv \begin{vmatrix} 1 & \alpha + \alpha' & \alpha\alpha' \\ 1 & \beta + \beta' & \beta\beta' \\ 1 & \gamma + \gamma' & \gamma\gamma' \end{vmatrix}.$$

Multiplying the determinant by

$$\begin{vmatrix} \alpha^2 & -\alpha & 1 \\ \beta^2 & -\beta & 1 \\ \gamma^2 & -\gamma & 1 \end{vmatrix},$$

and then removing the factor $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ from both sides of the equation, the value of Δ is easily expressed as follows :—

$$\Delta \equiv (\alpha-\beta')(\beta-\gamma')(\gamma-\alpha') + (\alpha'-\beta)(\beta'-\gamma)(\gamma'-\alpha).$$

This result may also be derived from the determinant of Ex. 13, whose vanishing expresses the general homographic relation between two sets of four points.

16. Expand the determinant

$$\begin{vmatrix} x & 0 & 0 & 0 & a_4 \\ -1 & x & 0 & 0 & a_3 \\ 0 & -1 & x & 0 & a_2 \\ 0 & 0 & -1 & x & a_1 \\ 0 & 0 & 0 & -1 & a_0 \end{vmatrix}.$$

This is found to be identical with the quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4;$$

and it is easily seen that a polynomial of any degree can be expressed as a determinant of like form.

17. Prove

$$\begin{vmatrix} x & a_1 & a_2 & a_3 & 1 \\ \alpha & x & b_1 & b_2 & 1 \\ \alpha & \beta & x & c_1 & 1 \\ \alpha & \beta & \gamma & x & 1 \\ \alpha & \beta & \gamma & \delta & 1 \end{vmatrix} \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta);$$

a_1, a_2, a_3, b_1, b_2, c_1 being any quantities.

This follows by subtracting α times the last column from the first, β times the last from the second, &c. The student will have no difficulty in writing down the corresponding determinant of the $(n+1)^{th}$ order which is equal to the polynomial $f(x)$ whose roots are $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$.

18. Resolve into factors the determinant

$$\Delta \equiv \begin{vmatrix} (\alpha-\alpha')^2 & (\alpha-\beta')^2 & (\alpha-\gamma')^2 \\ (\beta-\alpha')^2 & (\beta-\beta')^2 & (\beta-\gamma')^2 \\ (\gamma-\alpha')^2 & (\gamma-\beta')^2 & (\gamma-\gamma')^2 \end{vmatrix}.$$

Here $$\Delta = \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \begin{vmatrix} 1 & -2\alpha' & \alpha'^2 \\ 1 & -2\beta' & \beta'^2 \\ 1 & -2\gamma' & \gamma'^2 \end{vmatrix};$$

and these two determinants may be resolved as in Ex. 9, Art. 122.

19. Resolve into factors the determinant

$$\Delta \equiv \begin{vmatrix} (\alpha-\alpha')^3 & (\alpha-\beta')^3 & (\alpha-\gamma')^3 \\ (\beta-\alpha')^3 & (\beta-\beta')^3 & (\beta-\gamma')^3 \\ (\gamma-\alpha')^3 & (\gamma-\beta')^3 & (\gamma-\gamma')^3 \end{vmatrix}.$$

Multiplying the two rectangular arrays

$$\left.\begin{matrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{matrix}\right\} (1). \qquad \left.\begin{matrix} 1 & -3\alpha' & 3\alpha'^2 & -\alpha'^3 \\ 1 & -3\beta' & 3\beta'^2 & -\beta'^3 \\ 1 & -3\gamma' & 3\gamma'^2 & -\gamma'^3 \end{matrix}\right\} (2),$$

Δ becomes equal to the sum of four terms, from each of which we can take out as a factor the product of the two determinants

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha' & \alpha'^2 \\ 1 & \beta' & \beta'^2 \\ 1 & \gamma' & \gamma'^2 \end{vmatrix}.$$

The remaining factor is

$$3\,\{3\alpha\beta\gamma - \Sigma\beta\gamma\,\Sigma\alpha' + \Sigma\beta'\gamma'\,\Sigma\alpha - 3\alpha'\beta'\gamma'\},$$

which can be written also in the form

$$3\{(\alpha-\alpha')(\beta-\beta')(\gamma-\gamma') + (\alpha-\beta')(\beta-\gamma')(\gamma-\alpha') + (\alpha-\gamma')(\beta-\alpha')(\gamma-\beta')\}.$$

20. Prove the expansion

$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left\{1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right\}.$$

This is easily proved by subtracting the first column from each of the others, and then expanding the determinant as a linear function of the constituents of the first column. It will be apparent from the nature of the proof that the value of the similar determinant of the n^{th} order is $a_1a_2a_3 \ldots a_n\left\{1 + \Sigma\frac{1}{a_1}\right\}$.

21. Prove the relation

$$\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x),$$

where $$f(x) \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta.$$

This can be derived from the previous example, or proved independently in a similar way. As in the last example, the determinant of this form of the n^{th} degree can be similarly expressed.

22. Each of the coefficients of any equation can be expressed in terms of the roots as the quotient of two determinants.

The student can easily extend to any degree the following application to the equation of the third degree.

From Ex. 10, Art. 122, we have

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} \equiv -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

Expanding the determinant, this identity can be written

$$\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} x^3 - \begin{vmatrix} \alpha^3 & \alpha & 1 \\ \beta^3 & \beta & 1 \\ \gamma^3 & \gamma & 1 \end{vmatrix} x^2 + \begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} x - \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha \\ \beta^3 & \beta^2 & \beta \\ \gamma^3 & \gamma^2 & \gamma \end{vmatrix}$$

$$\equiv \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \{x^3 - p_1x^2 + p_2x - p_3\},$$

from which the above proposition follows; p_1, p_2, p_3 being the coefficients of the equation whose roots are α, β, γ.

23. Express as a determinant the reducing cubic of a biquadratic.

Writing down the equations which result from the identity

$$(a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4) \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c'),$$

assuming $6a_0\phi \equiv ac' + a'c - 2bb'$, and substituting in the following identity:—

$$\begin{vmatrix} a & a' & 0 \\ & b' & 0 \\ c & c' & 0 \end{vmatrix} \times \begin{vmatrix} a' & a & 0 \\ b' & b & 0 \\ c' & c & 0 \end{vmatrix} \equiv \begin{vmatrix} 2aa' & ab' + a'b & ac' + a'c \\ ab' + a'b & 2bb' & bc' + b'c \\ ac' + a'c & bc' + b'c & 2cc' \end{vmatrix} = 0,$$

we easily find the equation

$$\begin{vmatrix} a_0 & a_1 & {}_2 + 2a_0\phi \\ a_1 & a_2 - a_0\phi & a_3 \\ a_2 + 2a_0\phi & a_3 & a_4 \end{vmatrix} = 0,$$

which when expanded is found to be identical with the standard reducing cubic.

24. Find the condition that a biquadratic should be capable of being expressed as the sum of two fourth powers; and, expressing it in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv l(x+\theta)^4 + m(x+\phi)^4,$$

find the quadratic whose roots are θ and ϕ.

From this identity we have the following equations:—

$$\left.\begin{aligned} l + m &= a, \\ l\theta + m\phi &= b, \\ l\theta^2 + m\phi^2 &= c, \\ l\theta^3 + m\phi^3 &= d, \\ l\theta^4 + m\phi^4 &= e. \end{aligned}\right\} \qquad (1)$$

Assuming $\lambda + \mu x + \nu x^2 = 0$ as the equation whose roots are θ and ϕ, we easily obtain the three equations

$$\lambda a + \mu b + \nu c = 0,$$

$$\lambda b + \mu c + \nu d = 0,$$

$$\lambda c + \mu d + \nu e = 0,$$

from which we have at once the required condition $J = 0$; and from the first two, along with the assumed equation, we obtain the following quadratic whose roots are θ and ϕ:—

$$\begin{vmatrix} 1 & x & x^2 \\ a & b & c \\ b & c & d \end{vmatrix} = 0.$$

If it were required to express a cubic as the sum of two cubes, in the form $l(x+\theta)^3 + m(x+\phi)^3$, the first four of the above equations (1) would lead to the same quadratic for θ and ϕ.

25. For the biquadratic

$$A(x+\alpha)^4 + B(x+\beta)^4 + C(x+\gamma)^4 + D(x+\delta)^4 = 0,$$

prove

$$H = \Sigma AB(\alpha - \beta)^2,$$

$$I = \Sigma AB(\alpha - \beta)^4,$$

$$J = \Sigma ABC(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2.$$

These expressions are true for a biquadratic written as the sum of any number of fourth powers. If it can be written as the sum of two only, $J = 0$, since only A and B remain; and if it reduces to one fourth power, H, I, J all vanish—results already obtained by other methods.

26. Discuss the determinant of the fourth order, whose constituents $(\alpha - \alpha')^4$, $(\alpha - \beta')^4$, &c. are arranged as in Ex. 19, p. 299; and if $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are the roots of two given biquadratic equations, show that the value in terms of the coefficients contains as a factor

$$ae' + a'e - 4(bd' + b'd) + 6cc'.$$

When the two biquadratics are identical, this factor becomes $2I$.

27. Find the condition that the homogeneous quadratic function of three variables

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

should be resolvable into two factors.

Equating the given function to the product of the factors

$$(\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z),$$

we readily find

$$\begin{vmatrix} \alpha & \alpha' & 0 \\ \beta & \beta' & 0 \\ \gamma & \gamma' & 0 \end{vmatrix} \begin{vmatrix} \alpha' & \alpha & 0 \\ \beta' & \beta & 0 \\ \gamma' & \gamma & 0 \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

hence the required condition is that the determinant last written should vanish.

28. Show that the most general values of x, y, z, w which satisfy the two homogeneous equations

$$ax + by + cz + dw = 0, \quad a'x + b'y + c'z + d'w = 0$$

may be expressed symmetrically in terms of two indeterminates X, Y in the form

$$(ab')(ac')(ad')x = aX + a'Y,$$

$$(ba')(bc')(bd')y = bX + b'Y, \text{ \&c.}$$

This can be proved by joining to the two given equations the two following:—

$$\frac{a^2}{a'}x + \frac{b^2}{b'}y + \frac{c^2}{c'}z + \frac{d^2}{d'}w = \lambda, \quad \frac{a'^2}{a}x + \frac{b'^2}{b}y + \frac{c'^2}{c}z + \frac{d'^2}{d}w = \mu,$$

where λ, μ are indeterminate quantities; by then solving for x, y, z, w, as in Art. 134, and reducing the determinants as in Ex. 12, p. 297; and finally making $X = a'b'c'd'\lambda$, $Y = -abcd\mu$.

29. If in any determinant r columns (or rows) become identical when $x = a$, then $(x - a)^{r-1}$ is a factor in the determinant.

This appears easily by subtracting in the given determinant one of the r columns from each of the others. The resulting $r - 1$ columns must each contain $x - a$ as a factor, since by hypothesis each constituent in it vanishes when $x = a$.

30. Find the value of the determinant of the n^{th} order

$$\Delta \equiv \begin{vmatrix} x & a & a & . & a \\ a & x & a & . & a \\ a & a & x & . & a \\ . & . & . & . & . \\ a & a & a & . & x \end{vmatrix},$$

whose leading constituents are all equal to x, and the remaining constituents all equal to a.

By the preceding example Δ must contain $(x-a)^{n-1}$ as a factor; and by adding all the columns we see that it must also contain $x+(n-1)\,a$ as a factor. Hence Δ can differ by a numerical factor only from the product of these; and by comparing the product with the leading term we find

$$\Delta = (x-a)^{n-1}\,\{x+(n-1)\,a\}.$$

This result can readily be proved directly without the aid of Ex. 29.

31. The determinant

$$\begin{vmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ f_1(\beta) & f_2(\beta) & f_3(\beta) \\ f_1(\gamma) & f_2(\gamma) & f_3(\gamma) \end{vmatrix},$$

in which f_1, f_2, f_3 are any rational integral functions, contains the difference-product $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ as a factor.

This appears readily by reasoning similar to that of Ex. 29. Determinants of this nature, in which the constituents of any column (or row) are functions of the same form, and the constituents of any row (or column) involve the same quantity, are called *alternants*. It is plain that the result is general, and that the alternant of any order contains as a factor the difference-product of all the quantities involved. The determinants of Exs. 9, 10, Art. 122, and Exs. 11, 12, Art. 130, are alternants of the simplest form.

32. Express in the form of a determinant the quotient of the alternant in the preceding example by the difference-product.

Assuming, to fix the ideas, that the functions involved are each of the fifth degree (which will include lower degrees by making some coefficients vanish), we may write

$$f_1(\alpha) \equiv a_1\alpha^5 + b_1\alpha^4 + c_1\alpha^3 + d_1\alpha^2 + e_1\alpha + f_1,$$

$$f_2(\alpha) \equiv a_2\alpha^5 + b_2\alpha^4 + c_2\alpha^3 + d_2\alpha^2 + e_2\alpha + f_2,$$

$$f_3(\alpha) \equiv a_3\alpha^5 + b_3\alpha^4 + c_3\alpha^3 + d_3\alpha^2 + e_3\alpha + f_3.$$

Now taking α, β, γ to be the roots of the equation

$$x^3 + px^2 + qx + r = 0,$$

and forming the product of the following determinants :—

$$\begin{vmatrix} \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^5 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & 1 & p & q & r \\ 0 & 1 & p & q & r & 0 \\ 1 & p & q & r & 0 & 0 \end{vmatrix},$$

it readily appears that the determinant last written is the required quotient.

A similar method may be used to form the quotient when the alternant is of any order, and f_1, f_2, f_3, &c., rational integral functions of any degrees.

33. Resolve the following determinant into linear factors:—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}.$$

In all the rows the constituents are the same five quantities taken in circular order, a different one standing first in each row. A determinant of this kind is called a *circulant*. It is convenient to write a circulant in the form here given, viz., such that the same constituent occupies the diagonal place throughout. Taking θ to be any root of the equation $x^5 - 1 = 0$, and adding to the first column the sum of the constituents of the remaining columns multiplied by θ, θ^2, θ^3, θ^4, respectively, we observe that the following are factors of the determinant:—

$$\begin{gathered} a_1 + a_2 + a_3 + a_4 + a_5, \\ a_1 + \theta a_2 + \theta^2 a_3 + \theta^3 a_4 + \theta^4 a_5, \\ a_1 + \theta^2 a_2 + \theta^4 a_3 + \theta a_4 + \theta^3 a_5, \\ a_1 + \theta^3 a_2 + \theta a_3 + \theta^4 a_4 + \theta^2 a_5, \\ a_1 + \theta^4 a_2 + \theta^3 a_3 + \theta^2 a_4 + \theta a_5, \end{gathered}$$

the five roots of $x^5 - 1 = 0$ being $1, \theta, \theta^2, \theta^3, \theta^4$; and comparing the coefficient of a_1^5 in both expressions it appears that the numerical factor is unity (cf. Ex. 13, Art. 120). A circulant of any order can be treated in a similar manner.

34. The product of two circulants of the same order is a circulant.

35. Calculate the determinant of the n^{th} order

$$\Delta_n \equiv \begin{vmatrix} a_n & b_n & 0 & 0 & 0 & . \\ -1 & a_{n-1} & b_{n-1} & 0 & 0 & . \\ 0 & -1 & a_{n-2} & b_{n-2} & 0 & . \\ 0 & 0 & -1 & a_{n-3} & b_{n-3} & . \\ . & . & . & . & . & . \end{vmatrix},$$

in which all the constituents are zero except those which lie in the diagonal and in lines adjacent to it on either side and parallel to it, one of these latter sets consisting of constituents each equal to -1.

Expanding in terms of the first column, we have the following relation connecting three determinants of the kind here considered whose orders are n, $n-1$, $n-2$:—

$$\Delta_n = a_n \Delta_{n-1} + b_n \Delta_{n-2}.$$

By aid of this equation the calculation of any determinant is reduced to that of the two next inferior to it in the series Δ_n, Δ_{n-1}, Δ_{n-2}, . . . Δ_2, Δ_1; and the values of Δ_1 and Δ_2 are plainly a_1 and $a_2a_1 + b_2$, respectively.

Dividing the equation just given by Δ_{n-1} we have

$$\frac{\Delta_n}{\Delta_{n-1}} = a_n + \frac{b_n}{\dfrac{\Delta_{n-1}}{\Delta_{n-2}}};$$

replacing by a similar value the quotient of Δ_{n-1} by Δ_{n-2}, and continuing the process, it appears that the quotient of any determinant by the one next below it in the series can be expressed as a continued fraction in terms of the given constituents. On account of this property determinants of the form here treated are called *continuants*. When each of the constituents b_n, b_{n-1}, . . . b_3, b_2 (in the line above the diagonal) is equal to + 1 the resulting determinant is a *simple continuant*.

36. Calculate the determinant of the n^{th} order

$$\Delta_n \equiv \begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & . \\ \beta & \alpha & 1 & 0 & 0 & . \\ 0 & \beta & \alpha & 1 & 0 & . \\ 0 & 0 & \beta & \alpha & 1 & . \\ . & . & . & . & . & . \end{vmatrix},$$

whose only constituents which do not vanish are α, β, 1, occupying the diagonal and the lines adjacent and parallel to it as here represented.

The calculation is readily effected for any particular value of n, in a manner similar to that of the last example, by aid of the equation

$$\Delta_n = \alpha\Delta_{n-1} - \beta\Delta_{n-2},$$

the values of Δ_1 and Δ_2 being α and $\alpha^2 - \beta$, respectively.

By examining the formation of the successive values of Δ, the student will readily observe that the terms contained in the result are

$$\alpha^{2r}, \quad \alpha^{2r-2}\beta, \quad \alpha^{2r-4}\beta^2, \ . \ . \ . \ \alpha^2\beta^{r-1}, \quad \beta^r,$$

when n is even and of the form $2r$; and

$$\alpha^{2r+1}, \quad \alpha^{2r-1}\beta, \quad \alpha^{2r-3}\beta^2, \ . \ . \ . \ \alpha^3\beta^{r-1}, \quad \alpha\beta^r,$$

when n is odd and of the form $2r + 1$.

For the purposes of a subsequent investigation, in which the results just stated will be made use of, it is not necessary to know the general forms of the numerical coefficients which enter into these expressions; but such forms can be arrived at without difficulty, and the following general expression obtained for Δ_n:—

$$\Delta_n = \alpha^n - (n-1)\,\alpha^{n-2}\beta + \frac{(n-3)(n-2)}{1\,.\,2}\,\alpha^{n-4}\beta^2 - \frac{(n-5)\,(n-4)\,(n-3)}{1\,.\,2\,.\,3}\,\alpha^{n-6}\beta^3 + \&c.$$

37. When a polynomial U is divided by another U' of lower dimensions, the coefficients of the quotient, and of the remainder, can be expressed as determinants in terms of the coefficients of U and U'.

The method employed in the following particular case is equally applicable in general. Let U be of the fifth, and U' of the third degree; the quotient and remainder can then be represented as follows:—

$$Q \equiv q_0x^2 + q_1x + q_2, \quad R \equiv r_0x^2 + r_1x + r_2.$$

Also, let

$$U \equiv a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5, \quad U' \equiv a_0'x^3 + a_1'x^2 + a_2'x + a_3'.$$

From the identity $\qquad U \equiv QU' + R,$

we have the following equations :—

$$\begin{aligned}
a_0 &= q_0a_0', \\
a_1 &= q_0a_1' + q_1a_0', \\
a_2 &= q_0a_2' + q_1a_1' + q_2a_0', \\
a_3 &= q_0a_3' + q_1a_2' + q_2a_1' + r_0, \\
a_4 &= \qquad\quad q_1a_3' + q_2a_2' + r_1, \\
a_5 &= \qquad\qquad\qquad q_2a_3' + r_2.
\end{aligned}$$

Solving by Art. 134; q_0, q_1, q_2 are expressed as determinants by means of the first three of these equations; and taking the first three with each of the others in succession, we determine r_0, r_1, r_2. For example, to find r_0 we have, from the first four equations,

$$\begin{vmatrix} a_0' & 0 & 0 & -a_0 \\ a_1' & a_0' & 0 & -a_1 \\ a_2' & a_1' & a_0' & -a_2 \\ a_3' & a_2' & a_1' & -a_3 + r_0 \end{vmatrix} = 0, \quad \text{or} \quad a_0'^3 r_0 = \begin{vmatrix} a_0' & 0 & 0 & a_0 \\ a_1' & a_0' & 0 & a_1 \\ a_2' & a_1' & a_0' & a_2 \\ a_3' & a_2' & a_1' & a_3 \end{vmatrix}.$$

38. Find the general forms of the coefficients of the quotient, and of the remainder, when a polynomial of even degree $2m$ is divided by a quadratic.

Taking $x^2 + \alpha x + \beta$ as the given quadratic function, we have the identity

$$a_0x^{2m} + a_1x^{2m-1} + a_2x^{2m-2} + \ldots + a_{2m-2}x^2 + a_{2m-1}x + a_{2m}$$

$$\equiv (q_0x^{2m-2} + q_1x^{2m-3} + \ldots + q_{2m-3}x + q_{2m-2})(x^2 + \alpha x + \beta) + r_0x + r_1.$$

Writing down the first $r + 1$ equations, formed as in the preceding example, to solve for $q_0, q_1, q_2, \ldots q_r$, it is easily seen that the value of q_r thence derived is a determinant of the r^{th} order of the form treated in Ex. 36, bordered at the top with the constituents $1, 0, \ldots 0, a_0$, and at the right-hand side with $a_0, a_1, \ldots a_r$. Expanding this determinant in terms of the last column, it is immediately seen that any quotient is expressed by means of a series of the determinants of Ex. 36 in the form

$$q_r = a_r - a_{r-1}\Delta_1 + a_{r-2}\Delta_2 - \&c. \ldots \mp a_1\Delta_{r-1} \pm \Delta_r;$$

the upper or lower sign to be used according as r is even or odd. To obtain the coefficients of the remainder, we have the equations

$$\beta q_{2m-3} + a q_{2m-2} + r_0 = a_{2m-1},$$
$$\beta q_{2m-2} + r_1 = a_{2m}.$$

Expressing the values of q_{2m-3}, q_{2m-2} by the formula just proved, and attending to the results of Ex. 36, we derive the following general forms for r_0 and r_1 :—

$$r_0 = A_{2m-1} + A_{2m-3}\beta + A_{2m-5}\beta^2 + \ldots + A_3\beta^{m-2} + A_1\beta^{m-1},$$
$$r_1 = a_{2m} \quad + B_{2m-2}\beta + B_{2m-4}\beta^2 + \ldots + B_2\beta^{m-1} + B_0\beta^m,$$

in which the coefficients A, B are all functions of α, the highest power of α in any coefficient A or B being represented by the suffix attached to the coefficient.

39. If the leading constituents of a symmetric determinant be all increased by the same quantity x, the equation in x, obtained by equating to zero the determinant so formed, has all its roots real.

Let the determinant of the n^{th} order under consideration be denoted by Δ, and written in the form

$$\Delta_n \equiv \begin{vmatrix} a+x & h & g & . \\ h & b+x & f & . \\ g & f & c+x & . \\ . & . & . & . \end{vmatrix}.$$

Let the determinant obtained from this by erasing the first row and first column, *i.e.* the leading first minor of Δ_n, be denoted by Δ_{n-1} ; again, the leading first minor of Δ_{n-1} by Δ_{n-2} ; and so on, the last function Δ_1 obtained in this way being of the form $l + x$. To these we add the positive constant $\Delta_0 = 1$, which may be regarded as completing the series of minors and obtained by the same process, since Δ_n is not altered by affixing a last row and a last column consisting entirely of zero-elements, with the exception of the constituent $+ 1$ in the leading diagonal. We have now a series of $n + 1$ functions—

$$\Delta_n, \; \Delta_{n-1}, \; \Delta_{n-2}, \; \ldots \; \Delta_2, \; \Delta_1, \; \Delta_0,$$

whose degrees in x are represented by the suffixes. When $+\infty$ is substituted for x the signs are all positive, and when $-\infty$ is substituted, the signs (beginning with Δ_0) are alternately positive and negative. Hence, if x be regarded as increasing continuously, n changes of sign must be lost in this series during the passage from $-\infty$ to $+\infty$. Now it appears by the theorem of Art. 139, that a value of x which causes any function (excluding Δ_n, Δ_0) in this series to vanish gives opposite signs to the functions adjacent to it on either side. Δ_0 retains its sign throughout. It follows, exactly as in (2), Art. 89, that a change of sign can never be lost except when x passes through a real root of $\Delta_n = 0$. There must, therefore, exist n real roots of this equation in order that n changes may be lost during the passage of x from $-\infty$ to $+\infty$.

Any equation in the series, being of the same form as $\Delta_n = 0$, has all its roots real. It is plain also that each of these equations is a limiting equation (see Art. 83) with reference to the equation next above it in the series; since, in order that a change of sign may be lost between Δ_n and Δ_{n-1} at the passage through each of two consecutive roots of the former, the value of Δ_{n-1} must change sign between these two values of x. The equation $\Delta_n = 0$ may have equal roots, and by what has been just proved it appears that when this equation has r roots equal to a, the equation $\Delta_{n-1} = 0$ has $r - 1$ roots equal to a, the equation $\Delta_{n-2} = 0$ has $r - 2$ roots equal to a, and so on.

The determinant here discussed occurs in several investigations in pure mathematics and physics. The proof here given of the property above stated is taken from Salmon's *Higher Algebra* (Art. 46), to which work the student is referred for other proofs of the same theorem.

40. If the determinant of the preceding example have r roots equal to a; prove that every first minor has $r - 1$ roots equal to a; every second minor $r - 2$ roots equal to a, and so on.

Employing the notation A, H, G, . . . for the elements of the reciprocal determinant, we have the equation

$$AB - H^2 = \Delta_{n-2}\Delta_n.$$

Now it is easily seen by proper transpositions of rows and columns that every *leading* first minor contains the multiple root $r - 1$ times. It follows from the equation just written that the minor H must contain this root $r - 1$ times; and H may be taken to represent any first minor.

41. Find the conditions that the equation

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} = 0$$

should have equal roots.

Since each first minor must contain the double root, we readily derive the required conditions in the following form:—

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}.$$

[This and the preceding example are taken from Routh's *Dynamics of a System of Rigid Bodies*, Part II., Art. 61.]

42. Any symmetrical determinant can be altered so as to have any selected pair of conjugate constituents each zero, the determinant remaining symmetrical.

Consider, for example, the determinant obtained by putting $x = 0$ in the preceding example, and suppose it is required to remove the constituent g. Multiply

each constituent of the third column by a (dividing the whole determinant by a at the same time), and subtract from the constituents so altered those of the first column multiplied by g. Treat now the two corresponding rows in the same way; the resulting determinant is symmetrical, and in it g is replaced by zero. This process may be applied to a determinant of any order, to remove in succession all the conjugate constituents of the first row and column, and afterwards of the remaining rows and columns, so as to reduce the determinant finally to one, all of whose constituents vanish except those in the leading diagonal.

43. Reduce the following determinant, of any order, to a form in which x will appear in the leading constituents only :—

$$\begin{vmatrix} a_1x + a_1' & b_1x + b_1' & c_1x + c_1' & . \\ a_2x + a_2' & b_2x + b_2' & c_2x + c_2' & . \\ a_3x + a_3' & b_3x + b_3' & c_3x + c_3' & . \\ . & . & . & . \end{vmatrix}$$

Multiply by the determinant reciprocal to $(a_1 b_2 c_3 \ . \ . \ . \ l_n)$. If the given determinant is symmetrical, the determinant derived from it in this way will not be symmetrical; but a different process may be used to reduce it in that case to a *symmetrical* determinant which will have x present in the leading constituents only, viz. by removing the coefficients of x from all pairs of conjugate constituents in succession by a process exactly analogous to that of the preceding example. If the coefficients of x in the leading constituents of the reduced determinant should all have the same sign, it may be proved, just as in Ex. 39, that the corresponding equation will have all its roots real.

44. Let a determinant of the n^{th} order be divided into two rectangular arrays, one containing μ rows, and the other ν rows (where $\mu + \nu = n$), and let $\mu\nu$ sums of products be formed by operating with one array on the other as in the multiplication of determinants; if then such relations exist among the constituents that all these sums of products separately vanish, the determinants of order μ formed from the first array are proportional to determinants of order ν formed from the complementary constituents of the second.

To fix the ideas, we take a determinant of the fifth order, but the mode of proof is perfectly general. Let the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{vmatrix}$$

be split horizontally into two arrays, one of three, and the other of two rows; and let the following six relations exist:—

$$\Sigma a_1x_1 = 0, \quad \Sigma a_1y_1 = 0, \quad \Sigma b_1x_1 = 0, \quad \Sigma b_1y_1 = 0, \quad \Sigma c_1x_1 = 0, \quad \Sigma c_1y_1 = 0.$$

If now Δ be expanded by Laplace's theorem, and the minor determinants so taken (as can readily be done) that the expansion is written *with all positive signs*, e. g. in the form :—

$$\Delta \equiv (a_1b_2c_3)(x_4y_5) + (a_1b_3c_4)(x_2y_5) + (a_1b_2c_4)(x_5y_3) + (a_1b_2c_5)(x_3y_4) + \&c.,$$

it is proposed to prove that each minor determinant of the third order formed from the first array is proportional to its factor in the expansion of Δ so written.

We use for convenience the following notation for the expansion last written—

$$\Delta \equiv LL' + MM' + NN' + PP' + \&c.$$

Squaring the determinant Δ, making use of the above relations, replacing by their values the determinants obtained by squaring separately each of the component arrays, and equating the two values of Δ^2 thus obtained, we have

$$(LL' + MM' + NN' + \&c. \ldots)^2 = (L^2 + M^2 + N^2 + \&c. \ldots)(L'^2 + M'^2 + N'^2 + \&c. \ldots),$$

whence

$$(LM' - L'M)^2 + (LN' - L'N)^2 + (MN' - M'N)^2 + \&c. \ldots = 0,$$

from which we have at once

$$\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'} = \frac{P}{P'} = \&c.$$

45. Write down the relations which exist among the minors of the second order formed from a determinant of the fourth order divided equally into two rectangular arrays in the manner of the last example, like conditions being fulfilled.

We take the general determinant of the fourth order

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

and first expand it by Laplace's theorem. As the expansion of such a determinant in terms of its second minors is often required in practice, the student is recommended to accustom himself to write it with all positive signs as follows :—

$$(b_1c_2)(a_3d_4) + (c_1a_2)(b_3d_4) + (a_1b_2)(c_3d_4)$$
$$+ (a_1d_2)(b_3c_4) + (b_1d_2)(c_3a_4) + (c_1d_2)(a_3b_4).$$

The method of writing this down is obvious, the same arrangement being observed as on all former occasions where four letters were involved (cf. Ex. 17, Art. 27).

By the preceding example, we have at once the relations

$$\frac{(b_1c_2)}{(a_3d_4)} = \frac{(c_1a_2)}{(b_3d_4)} = \frac{(a_1b_2)}{(c_3d_4)} = \frac{(a_1d_2)}{(b_3c_4)} = \frac{(b_1d_2)}{(c_3a_4)} = \frac{(c_1d_2)}{(a_3b_4)},$$

provided the following four equations hold :—

$$\Sigma a_1a_3 = 0, \quad \Sigma a_1a_4 = 0, \quad \Sigma a_2a_3 = 0, \quad \Sigma a_2a_4 = 0.$$

What is here proved has an important application in geometry of three dimensions, with reference to the six co-ordinates of a right line. (See Salmon's *Analytic Geometry of Three Dimensions*, 4th ed., Art. 57b.)

It may be remarked here that it will be found convenient to write uniformly with positive signs the expansion of a determinant of the third order, which occurs so often in practical questions. Taking, for example, the determinant obtained by erasing the last row and last column of Δ, we write its expansion as follows, the three letters being taken in circular order :—

$$(a_1b_2c_3) \equiv a_1(b_2c_3) + b_1(c_2a_3) + c_1(a_2b_3).$$

46. Derive the equations (3) of Art. 135, for obtaining the ratios of n variables from $n-1$ linear homogeneous equations, from the proposition of Ex. 44.

47. Express by determinants the values of the unknown quantities derived from a set of given linear equations by the *Method of Least Squares.*

The given equations, which are greater in number than the unknown quantities, are supposed to have been obtained as the result of observation or experiment; and the numerical coefficients which enter into them, being consequently liable to errors of observation, are not known with certainty. In such cases the most reliable values of the unknown quantities are obtained in the manner about to be explained by what is called the "method of least squares." Take, for example, five equations of the form $a_1x + b_1y + c_1z = m_1$, $a_2x + b_2y + c_2z = m_2$, &c., between three unknown quantities x, y, z. Multiply them respectively by a_1, a_2, a_3, a_4, a_5, and add; again by b_1, b_2, b_3, b_4, b_5, and add; and again by c_1, c_2, c_3, c_4, c_5, and add. In this way the following three equations are obtained :—

$$x\Sigma a_1^2 + y\Sigma a_1b_1 + z\Sigma a_1c_1 = \Sigma a_1m_1,$$
$$x\Sigma a_1b_1 + y\Sigma b_1^2 + z\Sigma b_1c_1 = \Sigma b_1m_1,$$
$$x\Sigma a_1c_1 + y\Sigma b_1c_1 + z\Sigma c_1^2 = \Sigma c_1m_1;$$

from which we have without difficulty

$$x = \frac{(a_1b_2c_3)(m_1b_2c_3) + (a_1b_2c_4)(m_1b_2c_4) + \ldots + (a_3b_4c_5)(m_3b_4c_5)}{(a_1b_2c_3)^2 + (a_1b_2c_4)^2 + \ldots + (a_3b_4c_5)^2},$$

with corresponding values for y and z, each of these values containing ten terms in the numerator and ten in the denominator.

48. Show that the value of x given in the preceding example can be obtained by first eliminating y and z from every set of three of the five given equations, and then applying the method of least squares to the ten equations in x alone which result from the elimination.

CHAPTER XIII.

SYMMETRIC FUNCTIONS OF THE ROOTS.

140. **Newton's Theorem on the Sums of the Powers of the Roots.**—We now resume the discussion of symmetric functions of the roots of an equation, of which a short account has been previously given (see Art. 27); and proceed to prove certain general propositions relating to these functions:—

Prop. I.—*The sums of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients.*

Let the equation be

$$\begin{aligned} f(x) &\equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n \\ &\equiv (x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n) = 0. \end{aligned}$$

We proceed to calculate Σa^2, Σa^3, ... Σa^m; or, adopting the usual notation, $s_2, s_3, \ldots s_m$, in terms of the coefficients $p_1, p_2, \ldots p_n$.

We have, by Art. 72,

$$f'(x) = \frac{f(x)}{x - a_1} + \frac{f(x)}{x - a_2} + \ldots + \frac{f(x)}{x - a_n}$$

$$\equiv nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \ldots + 2p_{n-2}x + p_{n-1};$$

and we find, dividing by the method of Art. 8,

$$\frac{f(x)}{x - a} = \begin{array}{r|r|r|r} x^{n-1} + a & x^{n-2} + a^2 & x^{n-3} + a^3 & x^{n-4} + \ldots + a^{n-1} \\ + p_1 & + p_1 a & + p_1 a^2 & + p_1 a^{n-2} \\ & + p_2 & + p_2 a & + p_2 a^{n-3} \\ & & + p_3 & + \ldots \\ & & & + p_{n-2}\, a \\ & & & + p_{n-1}. \end{array}$$

If in this equation we replace a by each of the quantities $a_1, a_2, \ldots a_n$ in succession, and put $s_p = \Sigma a^p = a_1^p + a_2^p + \ldots + a_n^p$, we have, by adding all these results, the following value for $f'(x)$:—

$$
f'(x) = \begin{array}{r|r|r|r}
nx^{n-1} + s_1 & x^{n-2} + s_2 & x^{n-3} + s_3 & x^{n-4} + \ldots + s_{n-1} \\
+ np_1 & + p_1 s_1 & + p_1 s_2 & + p_1 s_{n-2} \\
 & + np_2 & + p_2 s_1 & + p_2 s_{n-3} \\
 & & + np_3 & \ldots \\
 & & & + p_{n-2} s_1 \\
 & & & + np_{n-1};
\end{array}
$$

whence, comparing this value of $f'(x)$ with the former, we obtain the following relations:—

$$
\begin{aligned}
& s_1 + p_1 = 0, \\
& s_2 + p_1 s_1 + 2p_2 = 0, \\
& s_3 + p_1 s_2 + p_2 s_1 + 3p_3 = 0, \\
& s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0, \\
& \ldots \quad \ldots \quad \ldots \\
& s_{n-1} + p_1 s_{n-2} + p_2 s_{n-3} + \ldots + p_{n-2} s_1 + (n-1) p_{n-1} = 0.
\end{aligned}
$$

The first equation determines s_1 in terms of $p_1, p_2, \ldots p_n$; the second s_2; the third s_3; and so on, until s_{n-1} is determined. We find in this way

$$
\begin{aligned}
& s_1 = -p_1, \quad s_2 = p_1^2 - 2p_2, \quad s_3 = -p_1^3 + 3p_1p_2 - 3p_3, \\
& s_4 = \ \ p_1^4 - 4p_1^2p_2 + 4p_1p_3 + 2p_2^2 - 4p_4, \\
& s_5 = -p_1^5 + 5p_1^3p_2 - 5p_1^2p_3 - 5(p_2^2 - p_4)p_1 + 5(p_2p_3 - p_5); \ \&c.
\end{aligned}
$$

Having shown how $s_1, s_2, s_3, \ldots s_{n-1}$ can be calculated in terms of the coefficients, we proceed now to extend our results

to the sums of all positive powers of the roots, viz. $s_n, s_{n+1}, \ldots s_m$. For this purpose we have

$$x^{m-n}f(x) \equiv x^m + p_1x^{m-1} + p_2x^{m-2} + \ldots + p_nx^{m-n}.$$

Replacing, in this identity, x by the roots $a_1, a_2, \ldots a_n$, in succession, and adding, we have

$$s_m + p_1s_{m-1} + p_2s_{m-2} + \ldots + p_ns_{m-n} = 0.$$

Now, giving m the values n, $n+1$, $n+2$, &c., successively, and observing that $s_0 = n$, we obtain from the last equation

$$\begin{aligned} s_n + p_1s_{n-1} + p_2s_{n-2} + \ldots + np_n &= 0, \\ s_{n+1} + p_1s_n + p_2s_{n-1} + \ldots + p_ns_1 &= 0, \\ s_{n+2} + p_1s_{n+1} + p_2s_n + \ldots + p_ns_2 &= 0, \text{ \&c.} \end{aligned}$$

Hence the sums of all positive powers of the roots may be expressed by integral functions of the coefficients. And by transforming the equation into one whose roots are the reciprocals of $a_1, a_2, a_3, \ldots a_n$, and applying the above formulas, we may express similarly all negative powers of the roots.

141. PROP. II.—*Every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients.*

It is sufficient to prove this theorem for integral functions only, since fractional symmetric functions can be reduced to a single fraction whose numerator and denominator are both integral symmetric functions. Every integral function of a_1, $a_2, a_3 \ldots . a_n$ is the sum of a number of terms of the form $Na_1^p\, a_2^q\, a_3^r \ldots$, where N is a numerical constant; and if this function be symmetrical we can write it under the form $N\Sigma a_1^p\, a_2^q\, a_3^r \ldots$, all the terms being of the same type. Therefore, if we prove that this quantity can be expressed rationally in terms of the coefficients, the theorem will be demonstrated. We shall first establish the following value of the symmetric function $\Sigma a_1^p a_2^q$:—

$$\Sigma a_1^p a_2^q = s_ps_q - s_{p+q}. \qquad (1)$$

To prove this, we multiply together s_p and s_q, where

$$s_p = a_1^p + a_2^p + a_3^p + \ldots a_n^p,$$

$$s_q = a_1^q + a_2^q + a_3^q + \ldots a_n^q;$$

whence

$$s_p s_q = a_1^{p+q} + a_2^{p+q} + \ldots + a_n^{p+q} + a_1^p a_2^q + a_1^q a_2^p + \&c.,$$

or

$$s_p s_q = s_{p+q} + \Sigma a_1^p a_2^q,$$

which expresses the double function $\Sigma a_1^p a_2^q$ in terms of the single functions s_p, s_q, s_{p+q}, in the form above written.

We proceed now to prove a similar expression for the triple function, i.e.,

$$\Sigma a_1^p a_2^q a_3^r = s_p s_q s_r - s_{q+r} s_p - s_{r+p} s_q - s_{p+q} s_r + 2 s_{p+q+r}. \quad (2)$$

Multiplying together $\Sigma a_1^p a_2^q$ and s_r, where

$$\Sigma a_1^p a_2^q = a_1^p a_2^q + a_1^q a_2^p + a_1^p a_3^q + \ldots$$

$$s_r = a_1^r + a_2^r + a_3^r + \ldots + a_n^r,$$

we obtain an expression consisting of three different parts, viz. terms of the form $\Sigma a_1^{p+r} a_2^q$, $\Sigma a_1^{q+r} a_2^p$, and $\Sigma a_1^p a_2^q a_3^r$.

Hence

$$s_r \Sigma a_1^p a_2^q = \Sigma a_1^{p+r} a_2^q + \Sigma a_1^{q+r} a_2^p + \Sigma a_1^p a_2^q a_3^r,$$

a formula connecting double and triple symmetric functions.

But, by (1),

$$\Sigma a_1^{p+r} a_2^q = s_{p+r} s_q - s_{p+q+r},$$

$$\Sigma a_1^{q+r} a_2^p = s_{q+r} s_p - s_{p+q+r},$$

$$\Sigma a_1^p a_2^q \ = s_p s_q \ - s_{p+q}.$$

Substituting these values, we find the triple function $\Sigma a_1^p a_2^q a_3^r$ expressed as above in terms of single functions in the series s_1, s_2, s_3, &c.

In the same manner the quadruple function $\Sigma a_1^p a_2^q a_3^r a_4^s$

can be made to depend on the triple function $\Sigma a_1^p a_2^q a_3^r$, and ultimately on s_1, s_2, s_3, &c.; and so on. Whence, finally, every rational symmetric function of the roots may be expressed in terms of the coefficients, since, by Prop. I., s_1, s_2, s_3, &c., can be so expressed.

The formulas (1) and (2) require to be modified when any of the exponents become equal.

Thus, if $p = q$, $a_1^p a_2^q = a_2^p a_1^q$, and the terms in (1) become equal two and two; therefore $\Sigma a_1^p a_2^q = 2\Sigma a_1^p a_2^p$; whence

$$\Sigma a_1^p a_2^p = \tfrac{1}{2}\left(s_p^2 - s_{2p}\right).$$

Similarly, if $p = q = r$ in $\Sigma a_1^p a_2^q a_3^r$, the six terms obtained by interchanging the roots in $a_1^p a_2^q a_3^r$ become all equal; hence

$$\Sigma a_1^p a_2^p a_3^p = \frac{1}{2.3}\left(s_p^3 - 3s_p s_{2p} + 2s_{3p}\right).$$

And, in general, if t exponents become equal, each term is repeated $1.2.3\ldots t$ times.

EXAMPLES.

1. Prove

$$\Sigma a_1^p a_2^q a_3^r a_4^t = s_p s_q s_r s_t - \Sigma s_{p+q} s_r s_t + 2\Sigma s_{p+q+r} s_t + \Sigma s_{p+q} s_{r+t} - 6s_{p+q+r+t}.$$

2. Prove

$$24\Sigma a_1^m a_2^m a_3^m a_4^m = s_m^4 - 6s_m^2 s_{2m} + 8s_m s_{3m} + 3s_{2m}^2 - 6s_{4m}.$$

142. PROP. III.—*The value of s_r, expressed in terms of $p_1, p_2, \ldots p_n$, is the coefficient of y^r in the expansion, by ascending powers of y, of* $-r\log y^n f\left(\frac{1}{y}\right)$.

Since

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = (x - a_1)(x - a_2)\ldots(x - a_n),$$

putting $\frac{1}{y}$ for x in this identical equation, we find

$$1 + p_1y + p_2y^2 + p_3y^3 + \ldots + p_ny^n = (1 - a_1y)(1 - a_2y)\ldots(1 - a_ny).$$

Now, taking the Napierian logarithms of both sides,

$$\begin{array}{l|l|l|l|l}
p_1 y + p_2 & y^2 + p_3 & y^3 + p_4 & y^4 + p_5 & y^5 + \&c. \dots + P_r y^r + \&c. \\
\quad -\frac{1}{2} p_1^2 & \quad -p_1 p_2 & \quad -p_1 p_3 & \quad -p_2 p_3 & \\
 & \quad +\frac{1}{3} p_1^3 & \quad -\frac{1}{2} p_2^2 & \quad +p_1 p_2^2 & \\
 & & \quad +p_1^2 p_2 & \quad +p_1^2 p_3 & \\
 & & \quad -\frac{1}{4} p_1^4 & \quad -p_1 p_4 & \\
 & & & \quad -p_1^3 p_2 & \\
 & & & \quad +\frac{1}{5} p_1^5 &
\end{array}$$

$$= -y s_1 - \frac{1}{2} y^2 s_2 - \frac{1}{3} y^3 s_3 - \dots - \frac{1}{r} y^r s_r - \&c.$$

Therefore, equating coefficients of y^r in both expansions,

$$s_r = -r P_r,$$

where P_r is the coefficient of y_r in $\log y^n f\left(\frac{1}{y}\right)$.

From the above identical equation it may be seen that s_r (r less than n) involves the coefficients $p_1, p_2, p_3, \dots p_r$ only; and, therefore, $p_{r+1}, p_{r+2}, \dots p_n$ may be made to vanish without affecting the form of the expression of s_r in terms of the coefficients.

143. *To express the coefficients in terms of the sums of the powers of the roots.*

Since

$$1 + p_1 y + p_2 y^2 + \dots + p_n y^n = (1 - a_1 y)(1 - a_2 y) \dots (1 - a_n y),$$

we have

$$\log(1 + p_1 y + \dots + p_n y^n) = -y s_1 - \frac{1}{2} y^2 s_2 - \dots - \frac{1}{r} y^r s_r - \dots; \quad (1)$$

and, therefore,

$$1 + p_1 y + p_2 y^2 + \dots + p_n y^n = e^{-y s_1 - \frac{1}{2} y^2 s_2 - \frac{1}{3} y^3 s_3 - \dots},$$

which becomes by expansion

$$\begin{array}{l|l|l|l}
1 - s_1 y - \frac{1}{2} s_2 & y^2 - \frac{1}{3} s_3 & y^3 - \frac{1}{4} s_4 & y^4 - \ldots \\
\quad + \frac{1}{1\,.\,2} s_1^2 & \quad + \frac{1}{1\,.\,2} s_1 s_2 & \quad + \frac{1}{3} s_1 s_3 & \\
 & \quad - \frac{1}{1\,.\,2\,.\,3} s_1^3 & \quad - \frac{1}{4} s_1^2 s_2 & \\
 & & \quad + \frac{1}{2\,.\,4} s_2^2 & \\
 & & \quad + \frac{1}{2\,.\,3\,.\,4} s_1^4 &
\end{array}$$

Now, comparing the coefficients of the different powers of y, we obtain values for $p_1, p_2, p_3, \ldots p_n$, in terms of $s_1, s_2, \ldots s_n$; and we see that p_r involves no sum of powers beyond s_r.

If the identity (1) be differentiated with regard to y, the equations of Art. 140 connecting the coefficients and sums of powers may be derived immediately from the resulting identity.

It is important to observe that the problem to express any symmetric function of the roots in terms of the coefficients or any coefficient in terms of the sums of the powers of the roots is perfectly definite, there being only one solution in each case.

General expressions, due to Waring, for s_m in terms of the coefficients, and for p_m in terms of the sums of the powers of the roots, will be found among the following examples which depend on the preceding propositions.

Examples.

1. Determine the value of

$$\phi(\alpha_1) + \phi(\alpha_2) + \ldots + \phi(\alpha_n),$$

where $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the roots of $f(x) = 0$, and $\phi(x)$ is any rational and integral function of x.

We have

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots + \frac{1}{x - \alpha_n},$$

and

$$\frac{f'(x)\,\phi(x)}{f(x)} = \frac{\phi(x)}{x - \alpha_1} + \frac{\phi(x)}{x - \alpha_2} + \ldots + \frac{\phi(x)}{x - \alpha_n}.$$

Performing the division, and retaining only the remainders on both sides of this equation, we have

$$\frac{R_0x^{n-1}+R_1x^{n-2}+\ldots+R_{n-1}}{f(x)}=\frac{\phi(a_1)}{x-a_1}+\frac{\phi(a_2)}{x-a_2}+\ldots+\frac{\phi(a_n)}{x-a_n};$$

whence

$$R_0x^{n-1}+R_1x^{n-2}+\ldots+R_{n-1}=\Sigma\phi(a_1)(x-a_2)(x-a_3)\ldots(x-a_n);$$

and, comparing the coefficients of x^{n-1} on both sides of this equation,

$$R_0=\Sigma\phi(a_1).$$

2. Prove that s_p is the coefficient of $\frac{1}{x^{p+1}}$ in the quotient of the division of $f'(x)$ by $f(x)$ arranged according to negative powers of x.

3. Prove that s_{-p} is the coefficient (with sign changed) of x^{p-1} in the same quotient arranged according to positive powers of x.

4. If the degree of $\phi(x)$ does not exceed $n-2$, prove

$$\sum_{r=1}^{r=n}\frac{\phi(a_r)}{f'(a_r)}=0,$$

where $\sum\limits_{r=1}^{r=n}$ denotes the sum obtained by giving r all values from 1 to n inclusive.

We have, by partial fractions,

$$\frac{\phi(x)}{f(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n};$$

and, multiplying across by $f(x)$, and putting x equal to a_1, a_2, . . . in succession,

$$\frac{\phi(x)}{f(x)}=\frac{\phi(a_1)}{f'(a_1)}\frac{1}{x-a_1}+\frac{\phi(a_2)}{f'(a_2)}\frac{1}{x-a_2}+\ldots+\frac{\phi(a_n)}{f'(a_n)}\frac{1}{x-a_n};$$

whence

$$\frac{x\phi(x)}{f(x)}=\sum_{r=1}^{r=n}\frac{\phi(a_r)}{f'(a_r)}\left(1+\frac{a_r}{x}+\frac{a_r^2}{x^2}+\ldots\right).$$

When $\phi(x)$ is of the degree $n-2$; expressing the first side of the equation as a function of $\frac{1}{x}$, it readily appears that there is no term without $\frac{1}{x}$ as a multiplier. We have, therefore, comparing coefficients,

$$\sum_{r=1}^{r=n}\frac{\phi(a_r)}{f'(a_r)}=0.$$

As ϕ may be any rational and integral function of degree not higher than $n-2$, we have the following particular cases which are worthy of special notice :—

$$\Sigma\frac{a^{n-2}}{f'(a)}=0,\quad \Sigma\frac{a^{n-3}}{f'(a)}=0,\ \ldots\ \Sigma\frac{a}{f'(a)}=0,\quad \Sigma\frac{1}{f'(a)}=0.$$

5. Given the following $n-2$ equations between n variables $x_1, x_2, \ldots x_n$:—

$$\sum_{r=1}^{r=n} x_r = 0, \quad \sum_{r=1}^{r=n} a_r x_r = 0, \ldots \sum_{r=1}^{r=n} a_r^{n-3} x_r = 0,$$

express the n variables in terms of two new variables X_1, X_2.

Ans. $x_r = \dfrac{X_1 + a_r X_2}{f'(a_r)}$.

6. Prove that the sum of all the homogeneous products Π_r, of the r^{th} degree, of the quantities $a_1, a_2, \ldots a_n$, is equal to

$$\Sigma \frac{a^{n+r-1}}{f'(a)}.$$

We have, putting $y = \dfrac{1}{x}$,

$$\frac{x^n}{f(x)} = \frac{1}{(1-a_1y)(1-a_2y)\ldots(1-a_ny)}$$

$$= (1 + a_1y + a_1^2y^2 + \ldots)(1 + a_2y + a_2^2y^2 + \ldots)\ldots(1 + a_ny + a_n^2y^2 + \ldots)$$

$$= 1 + \Pi_1 y + \Pi_2 y^2 + \ldots + \Pi_r y^r + \ldots$$

[The quantities $\Pi_1, \Pi_2, \ldots \Pi_r$, are defined by the identity last written. It will be observed that the coefficient of each symmetric function which enters into any one of them is unity.]

We have also
$$\frac{x^{n-1}}{f(x)} = \Sigma \frac{a^{n-1}}{f'(a)} \frac{1}{x-a},$$
and therefore
$$\frac{x^n}{f(x)} = \Sigma \frac{a^{n-1}}{f'(a)} \frac{1}{1-ay} = \Sigma \frac{a^{n+r-1}}{f'(a)} y^r;$$

whence, comparing coefficients of y^r in these two expansions,

$$\Pi_r = \Sigma \frac{a^{n+r-1}}{f'(a)}.$$

7. To express the coefficients of an equation in terms of the homogeneous products of the roots, and *vice versâ*.

From the equation of the preceding example

$$\frac{1}{(1-a_1y)(1-a_2y)\ldots(1-a_ny)} = 1 + \Pi_1y + \Pi_2y^2 + \ldots,$$

we have

$$(1 + p_1y + p_2y^2 + \ldots + p_ny^n)(1 + \Pi_1y + \Pi_2y^2 + \ldots) = 1,$$

which gives the following relations :—

$$p_1 + \Pi_1 = 0,$$

$$p_2 + \Pi_2 + p_1\Pi_1 = 0,$$

$$p_3 + \Pi_3 + p_1\Pi_2 + p_2\Pi_1 = 0, \text{ \&c.}$$

These equations (in which $p_1, p_2 \ldots p_n$, and $\Pi_1, \Pi_2, \ldots \Pi_n$ are interchangeable) determine $p_1, p_2 \ldots p_n$ in terms of $\Pi_1, \Pi_2, \ldots \Pi_n$, and *vice versa*.

By means of this and the preceding example the values of the following symmetric functions may be found in terms of the coefficients :—

$$\Sigma \frac{a^{n-1}}{f'(a)}, \quad \Sigma \frac{a^n}{f'(a)}, \quad \Sigma \frac{a^{n+1}}{f'(a)}, \text{ \&c.}$$

8. To express Π_r by the sums of the powers of the roots.

Representing by $\frac{1}{u}$ the product $(1 - a_1 y)(1 - a_2 y) \ldots (1 - a_n y)$, and differentiating, we find

$$\frac{1}{u}\frac{du}{dy} = \Sigma \frac{a}{1 - ay} = s_1 + s_2 y + s_3 y^2 + \ldots;$$

also $$u = 1 + \Pi_1 y + \Pi_2 y^2 + \ldots$$

We have, therefore,

$$(1 + \Pi_1 y + \Pi_2 y^2 + \ldots)(s_1 + s_2 y + s_3 y^2 + \ldots) = \Pi_1 + 2\Pi_2 y + 3\Pi_3 y^2 + \ldots$$

Now comparing the several coefficients of the different powers of y, we have a number of equations by means of which the sums of the homogeneous products $\Pi_1, \Pi_2, \Pi_3, \ldots$ may be expressed in terms of s_1, s_2, s_3, &c.

9. Prove the following formula for calculating the sums of the homogeneous products in terms of the coefficients :—

$$\frac{ds_{r+i}}{dp_r} = -(r + i)\Pi_i.$$

Differentiate both sides of equation (1) in Art. 143, and introduce Π_1, Π_2, &c., by the equation of Ex. 7.

10. To find a general expression for s_m in terms of the coefficients $p_1, p_2 \ldots p_n$, of an equation of the n^{th} degree.

We have

$$-\log_e(1 + p_1 y + p_2 y^2 + \ldots + p_n y_n) = \sum_{r=1}^{r=\infty} \frac{(-1)^r}{r}(p_1 y + p_2 y^2 + \ldots + p_n y^n)^r$$

$$= s_1 y + \frac{1}{2}s_2 y^2 + \frac{1}{3}s_3 y^3 + \ldots + \frac{1}{m}s_m y^m + \ldots$$

Now, making use of the known form of the coefficient of y^m in the expansion of $(p_1 y + p_2 y^2 + \ldots + p_n y^n)^r$ by the multinomial theorem, and comparing coefficients of y^m in the above equation, we find

$$s_m = \Sigma \frac{(-1)^r m\Gamma(r_1 + r_2 + \ldots + r_n)}{\Gamma(r_1 + 1)\Gamma(r_2 + 1) \ldots \Gamma(r_n + 1)} p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n},$$

in which

$$r_1 + r_2 + r_3 + \ldots + r_n = r,$$

$$r_1 + 2r_2 + 3r_3 + \ldots + nr_n = m;$$

and $r_1, r_2, r_3, \ldots r_n$ are to be given all positive integer values, zero included, which satisfy the last of these two equations. Also, representing by r_i any of these integers,

$$\Gamma(r_i + 1) = 1 . 2 . 3 \ldots r_i,$$

with the assumption that $\Gamma(1) = 1$ when $r_i = 0$.

11. To find a general expression for any coefficient p_m in terms of the sums of the powers of the roots $s_1, s_2, \ldots s_m$.

We have

$$1 + p_1y + p_2y^2 + \ldots + p_my^m + \ldots + p_ny^n = e^{-ys_1} \, . \, e^{-\frac{1}{2}y^2s_2} \, . \, e^{-\frac{1}{3}y^3s_3} \ldots$$

When the factors on the right-hand side of this equation are developed, and the coefficients of y^m on both sides compared, we find, employing the notation of the last example,

$$p_m = \sum \frac{(-1)^{r_1+r_2+\ldots+r_m} s_1^{r_1} s_2^{r_2} \ldots s_m^{r}}{\Gamma(r_1+1)\,\Gamma(r_2+1)\ldots\Gamma(r_m+1)\,2^{r_2}3^{r_3}\ldots m^{r_m}},$$

in which $r_1, r_2, \ldots r_m$ are to be given all positive values, zero included, which satisfy the equation

$$r_1 + 2r_2 + 3r_3 + \ldots + mr_m = m.$$

144. **Definitions. Theorem.**—The *weight* of any symmetric function of the roots is the degree in *all* the roots of any term in the function. For example, the weight of $\Sigma\alpha\beta^2\gamma^3$ is six.

The *order* of any symmetric function of the roots is the highest degree in which each root enters the function. For example, the order of $\Sigma\alpha\beta^2\gamma^3$ is three.

It has been proved (see Art. 28), that the weight of any symmetric function of the roots, when expressed by the coefficients $a_0, a_1, a_2, \ldots a_n$, is the same as the sum of the suffixes of each term in the expression. We now prove another important theorem, viz.:

If any symmetric function be expressed in terms of the coefficients $p_1, p_2, \ldots p_n$, *the degree in the coefficients is the same as the order of the symmetric function.* For example, $\Sigma\alpha^2\beta^2 = p_2^2 - 2p_1p_3 + 2p_4$, no term being of higher degree than the second in the coefficients, and the order of the symmetric function being two.

The student may easily satisfy himself of the truth of this theorem by observing that in the equations (2) of Art. 23, the value of each coefficient in terms of the roots contains each root in the first power only; hence the highest degree in the coefficients will be the same as the highest degree of the corresponding symmetric function in any individual root. We add the following formal proof, as it is in accordance with the proofs of certain general propositions to be given subsequently.

Replace the coefficients $p_1, p_2, \ldots p_n$ by $\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots \frac{a_n}{a_0}$.

Now, if $\phi(\alpha_1, \alpha_2, \ldots \alpha_n)$ denote any rational and integral symmetric function of the roots, we have

$$a_0^{\varpi}\phi(\alpha_1, \alpha_2, \ldots \alpha_n) = F(a_0, a_1, a_2, \ldots a_n),$$

where ϖ is the degree, in the coefficients, of $F(a_0, a_1, a_2, \ldots a_n)$, a homogeneous and integral function of the coefficients, not divisible by a_0.

We require now to show that ϖ is the order of ϕ. For this purpose change the roots into their reciprocals, and, therefore, $a_0, a_1, \ldots a_n$ into $a_n, a_{n-1}, \ldots a_0$. Whence

$$a_n^{\varpi}\phi\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots \frac{1}{\alpha_n}\right) = F(a_n, a_{n-1}, a_{n-2}, \ldots a_0)\,; \qquad (1)$$

also

$$\phi\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots \frac{1}{\alpha_n}\right) = \frac{\psi(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n)}{(\alpha_1\alpha_2\alpha_3 \ldots \alpha_n)^p},$$

where p is the order of ϕ, and ψ an integral function not divisible by the product of all the roots; $(\alpha_1\alpha_2\alpha_3 \ldots \alpha_n)^p$ being the lowest common denominator of all the terms. Substituting in (1), we have

$$a_0^p\,\psi(\alpha_1, \alpha_2, \ldots \alpha_n) = \pm\, a_n^{p-\varpi}\, F(a_n, a_{n-1}, \ldots a_0).$$

From this equation it follows that p is equal to ϖ; for if p were greater than ϖ, $\psi(\alpha_1, \alpha_2 \ldots \alpha_n)$ would be divisible by the product $\alpha_1\alpha_2 \ldots \alpha_n$, and if it were less, the function of the coefficients $F(a_n, a_{n-1}, \ldots a_0)$ would be divisible by a_n, both of which suppositions are contrary to hypothesis.

145. **Calculation of Symmetric Functions of the Roots.**—Any rational symmetric function can be calculated by the method of Art. 141. In practice, however, other methods are usually more convenient, as will appear from the examples given at the end of the present Article, and from the two following Articles, which contain propositions tending in many cases to facilitate the calculation of symmetric functions.

The number of terms in any symmetric function of the roots is easily determined. For example, the number of terms in $\Sigma a_1^3 a_2^2 a_3$ of the equation of the n^{th} degree is $n(n-1)(n-2)$, this being the number of permutations of n things taken three together. If the exponents of the roots in any term be not all different, the number of terms will be reduced. Thus $\Sigma \alpha^2 \beta\gamma$ for a biquadratic consists of twelve terms only (see Ex. 6, p. 48), and not of twenty-four, since the two permutations $\alpha\beta\gamma$, $\alpha\gamma\beta$ give only one distinct term, viz., $\alpha^2\beta\gamma$, in $\Sigma\alpha^2\beta\gamma$. The student acquainted with the theory of permutations will have no difficulty in effecting these reductions in any particular case. When two exponents of roots are equal, the number obtained on the supposition that they are all unequal is to be divided by 1.2; when three become equal this number is to be divided by $1.2.3$; and so on. In general, the number of terms in $\Sigma a_1^p a_2^q a_3^r \ldots$ of the equation of the n^{th} degree, each term containing m roots, and ν of the indices being equal, is

$$\frac{n(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots\nu}.$$

When the highest power in which any one root enters into the symmetric function of the roots is small, i.e. when the order of the function (see Art. 144) is low, the methods already illustrated in Art. 27 may be employed with advantage for the calculation of the symmetric function.

It is important to observe that when any symmetric function, whose degree in all the roots (i.e. its weight) is n, is calculated in terms of the coefficients $p_1, p_2 \ldots p_n$ for the equation of the n^{th} degree, its value for an equation of any higher degree (the numerical coefficients being all equal to unity) is precisely the same; for it is plain that no coefficient beyond p_n can enter into this value, and the equations of Art. 140, by means of which the calculation can be supposed to be made, have precisely the same form for an equation of the n^{th} degree as for equations of all higher degrees. It is also evident that the value of the same symmetric function for an equation of a degree m (lower than n)

is obtained by putting $p_{m+1}, p_{m+2}, \ldots p_n$ all equal to zero in the calculated value for an equation of the n^{th} degree, since the equation of lower degree can be derived from that of the n^{th} by putting the coefficients beyond p_m equal to zero; and the corresponding symmetric function reduces similarly by putting the roots $a_{m+1}, a_{m+2}, \ldots a_n$ each equal to zero.

EXAMPLES.

1. Calculate $\Sigma\alpha_1^2\alpha_2\alpha_3$ of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

Multiply together the equations

$$\Sigma\alpha_1 \qquad = -p_1,$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = -p_3.$$

In the product the term $\alpha_1^2\alpha_2\alpha_3$ occurs only once; the term $\alpha_1\alpha_2\alpha_3\alpha_4$ occurs four times, arising from the product of α_1 by $\alpha_2\alpha_3\alpha_4$, of α_2 by $\alpha_1\alpha_3\alpha_4$, of α_3 by $\alpha_1\alpha_2\alpha_4$, and of α_4 by $\alpha_1\alpha_2\alpha_3$. Hence

$$\Sigma\alpha_1^2\alpha_2\alpha_3 + 4\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = p_1p_3;$$

therefore

$$\Sigma\alpha_1^2\alpha_2\alpha_3 = p_1p_3 - 4p_4. \qquad \text{(Compare Ex. 6, Art. 27.)}$$

If the calculation were conducted by the method of Art. 141, we should have

$$\Sigma\alpha_1^2\alpha_2\alpha_3 = \tfrac{1}{2}s_2s_1^2 - s_1s_3 - \tfrac{1}{2}s_2^2 + s_4,$$

which leads, on substituting the values of Art. 140, to the same result; but it is evident that in this case the former process is much more simple, since the values of s_1, s_2, &c., introduce a number of terms which destroy one another.

2. Calculate $\Sigma\alpha_1^2\alpha_2^2$ for the general equation.

Squaring

$$\Sigma\alpha_1\alpha_2 = p_2,$$

we have

$$\Sigma\alpha_1^2\alpha_2^2 + 2\Sigma\alpha_1^2\alpha_2\alpha_3 + 6\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = p_2^2.$$

In squaring it is evident that the term $\alpha_1\alpha_2\alpha_3\alpha_4$ will arise from the product of $\alpha_1\alpha_2$ by $\alpha_3\alpha_4$, or of $\alpha_1\alpha_3$ by $\alpha_2\alpha_4$, or of $\alpha_1\alpha_4$ by $\alpha_2\alpha_3$; hence the coefficient of $\alpha_1\alpha_2\alpha_3\alpha_4$ in the result is 6, since each of these occurs twice in the square. The result differs from the similar equation of Ex. 8, Art. 27, only in having Σ before the term $\alpha_1\alpha_2\alpha_3\alpha_4$. Hence, finally,

$$\Sigma\alpha_1^2\alpha_2^2 = p_2^2 - 2p_1p_3 + 2p_4.$$

3. Calculate $\Sigma\alpha_1^3\alpha_2$ for the general equation.

We have, as in Ex. 9, Art. 27,

$$\Sigma\alpha_1^2\,\Sigma\alpha_1\alpha_2 = \Sigma\alpha_1^3\,\alpha_2 + \Sigma\alpha_1^2\,\alpha_2\,\alpha_3.$$

Hence, employing previous results,

$$\Sigma\alpha_1^3\,\alpha_2 = p_1^2\,p_2 - 2p_2^2 - p_1\,p_3 + 4p_4.$$

4. Calculate $\Sigma\alpha_1^2\,\alpha_2^2\,\alpha_3$ for the general equation.

The result will be the same as if the calculation were made for the equation of the fifth degree.

To obtain the symmetric function we multiply together $\Sigma\alpha_1\alpha_2$ and $\Sigma\alpha_1\alpha_2\alpha_3$; and consider what types of terms, involving the five roots α_1, α_2, α_3, α_4, α_5, can result. The term $\alpha_1^2\alpha_2^2\alpha_3$ will occur only once in the product, since it can only arise by multiplying $\alpha_1\alpha_2$ by $\alpha_1\alpha_2\alpha_3$. Terms of the type $\alpha_1^2\alpha_2\alpha_3\alpha_4$ will occur, each three times; since $\alpha_1^2\alpha_2\alpha_3\alpha_4$ will arise from the product of $\alpha_1\alpha_2$ by $\alpha_1\alpha_3\alpha_4$, of $\alpha_1\alpha_3$ by $\alpha_1\alpha_2\alpha_4$, or of $\alpha_1\alpha_4$ by $\alpha_1\alpha_2\alpha_3$; and it cannot arise in any other way. The term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ will occur ten times in the product, since it will arise from the product of any pair by the other three roots, and there are ten combinations in pairs of the five roots. We have, then, for the general equation,

$$\Sigma\alpha_1\alpha_2\,\Sigma\alpha_1\alpha_2\alpha_3 = \Sigma\alpha_1^2\alpha_2^2\alpha_3 + 3\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4 + 10\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.$$

[We can verify this equation when $n = 5$, just as in Ex. 9, Art. 27; for the product of two factors, each consisting of 10 terms, will contain 100 terms. These are made up of the 30 terms contained in $\Sigma\alpha_1^2\alpha_2^2\alpha_3$, along with the 20 terms contained in $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4$, each taken three times, and the term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ taken 10 times.]

Thus the calculation of the required symmetric function involves that of $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4$; for which we easily find

$$\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = \Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4 + 5\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.$$

Hence, finally, we obtain

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3 = -\,p_2\,p_3 + 3p_1\,p_4 - 5p_5.$$

The process of Art. 141 would involve the calculation of s_5; and many terms would be introduced through the values of s_1, s_2, &c., which disappear in the result.

5. Find the value of $\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4$ for the general equation.

We multiply together $\Sigma\alpha_1\alpha_2$ and $\Sigma\alpha_1\alpha_2\alpha_3\alpha_4$, and consider what types of terms can arise involving the six roots α_1, α_2, α_3, α_4, α_5, α_6. The term $\alpha_1^2\alpha_2^2\alpha_3\alpha_4$ can occur

Terms of the type $\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5$, will each occur four times, this term arising from the product of $\alpha_1\alpha_2$ by $\alpha_1\alpha_3\alpha_4\alpha_5$, or of $\alpha_1\alpha_3$ by $\alpha_1\alpha_2\alpha_4\alpha_5$, or of $\alpha_1\alpha_4$ by

$\alpha_1\alpha_2\alpha_3\alpha_5$, or of $\alpha_1\alpha_5$ by $\alpha_1\alpha_2\alpha_3\alpha_4$. The term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ will occur 15 times, this being the number of combinations in pairs of the six roots. Hence

$$\Sigma\alpha_1\alpha_2\,\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = \Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 + 4\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 15\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6.$$

We have again, for the calculation of $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5$,

$$\Sigma\alpha_1\,\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = \Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 6\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6.$$

Hence, finally,

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 = p_2p_4 - 4p_1p_5 + 9p_6.$$

6. Find the value of $\Sigma\alpha_1^2\alpha_2^2\alpha_3^2$ in terms of the coefficients of the general equation.

Here, squaring $\Sigma\alpha_1\alpha_2\alpha_3$, we have

$$\Sigma\alpha_1\alpha_2\alpha_3\,\Sigma\alpha_1\alpha_2\alpha_3 = \Sigma\alpha_1^2\alpha_2^2\alpha_3^2 + 2\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 + 6\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 20\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6,$$

from which we obtain

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3^2 = p_3^2 - 2p_2p_4 + 2p_1p_5 - 2p_6.$$

146. **Calculation of Symmetric Functions continued.** —By aid of the following differential equation, connecting a function of the coefficients and its value in terms of the sums of the powers, symmetric functions can often be calculated with great facility:—

$$\frac{d}{ds_r}F(p_1, p_2, \ldots p_n) = -\frac{1}{r}\left(\frac{dF}{dp_r} + p_1\frac{dF}{dp_{r+1}} + \ldots + p_{n-r}\frac{dF}{dp_n}\right).$$

To prove this equation, we take the equation (1) of Art. 143, and differentiate it with regard to s_r. Comparing coefficients of the different powers of y, we have

$$\frac{dp_q}{ds_r} = 0, \text{ when } q < r; \quad \frac{dp_r}{ds_r} = -\frac{1}{r}; \quad \frac{dp_{r+k}}{ds_r} = -\frac{1}{r}p_k;$$

and substituting these values in

$$\frac{d}{ds_r}F(p_1, p_2, \ldots p_n) = \frac{dF}{dp_1}\frac{dp_1}{ds_r} + \frac{dF}{dp_2}\frac{dp_2}{ds_r} + \ldots + \frac{dF}{dp_n}\frac{dp_n}{ds_r},$$

we have at once the equation above written.

Examples.

1. Calculate the value of the symmetric function $\Sigma\alpha_1^2\,\alpha_2^2\,\alpha_3^2\,\alpha_4^2$ of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0.$$

Knowing the order and weight of any symmetric function, we can write down the literal part of its value in terms of the coefficients. Here Σ is of the second order, and its weight is eight; hence

$$\Sigma = t_0 p_8 + t_1 p_7 p_1 + t_2 p_6 p_2 + t_3 p_5 p_3 + t_4 p_4^2,$$

where t_0, t_1, t_2, &c., are numerical coefficients to be determined.

Terms such as $p_6 p_1^2$, $p_5 p_1 p_2$, $p_5 p_1^3$, &c., although of the right weight, are of too high an order, and therefore cannot enter into the expression for Σ. Again, Σ expressed in terms of the sums of the powers of the roots is of the form $F(s_2, s_4, s_6, s_8)$; for, in general, $\Sigma\alpha_1^p\,\alpha_2^q\,\alpha_3^r \ldots$, expressed in terms of the sums of the powers of the roots, is made up of terms such as s_p, s_{p+q}, s_{p+q+r}, $\ldots$ s_{kp}, $\ldots$ all of which are sums of even powers when p, q, r, $\ldots$ are even; therefore in this case none but even sums of powers enter into the expression for Σ.

Also, since $\dfrac{d\Sigma}{ds_3} = 0$, and $\dfrac{d\Sigma}{ds_7} = 0$, we have, using the formula above given for $\dfrac{dF}{ds_r}$,

$$t_0 p_5 + t_1 p_1 p_4 + t_2 p_3 p_2 + t_3(p_2 p_3 + p_5) + 2t_4 p_1 p_4 = 0,$$
$$t_0 p_1 + t_1 p_1 = 0.$$

From these equations we infer

$$t_0 + t_1 = 0, \quad t_2 + t_3 = 0, \quad t_3 + t_0 = 0, \quad t_1 + 2t_4 = 0;$$

but $t_4 = 1$, since for a quartic $\Sigma = p_4^2$; therefore

$$t_1 = -2, \quad t_0 = 2, \quad t_3 = -2, \quad t_2 = 2;$$

and, substituting these values of t_0, t_1, t_2, t_3, t_4,

$$\Sigma\alpha_1^2\,\alpha_2^2\,\alpha_3^2\,\alpha_4^2 = 2p_8 - 2p_7 p_1 + 2p_6 p_2 - 2p_5 p_3 + p_4^2.$$

2. Calculate $\Sigma\alpha_1^2\,\alpha_2^2\,\alpha_3^2$ for the same equation.

Ans. $-2p_6 + 2p_1 p_5 - 2p_2 p_4 + p_3^2$. (Compare Ex. 6, Art. 145.)

3. Calculate for the same equation the symmetric function $\Sigma\alpha_1^3\,\alpha_2^2\,\alpha_3$.

Here the weight is six, and the order three; hence

$$\Sigma\alpha_1^3\,\alpha_2^2\,\alpha_3 = t_0 p_6 + t_1 p_5 p_1 + t_2 p_4 p_2 + t_3 p_4 p_1^2 + t_4 p_3^2 + t_5 p_1 p_2 p_3 + t_6 p_2^3.$$

Also Σ, expressed in terms of s_1, s_2, s_3, &c., is (see Art. 141),

$$s_1 s_2 s_3 - s_1 s_5 - s_3^2 - s_2 s_4 + 2s_6.$$

Now, differentiating these two values of Σ with regard to s_6, and comparing differential coefficients, we have

$$t_0 \frac{dp_6}{ds_6} = -\frac{t_0}{6} = 2, \quad \text{or} \quad t_0 = -12.$$

Differentiating with regard to s_5, we have

$$t_0 p_1 + t_1 p_1 = 5s_1 = -5p_1 ; \;\therefore\; t_1 = 7.$$

Differentiating with regard to s_4,

$$t_0 p^2 + t_1 p_1^2 + t_2 p_2 + t_3 p_1^2 = 4s_2 = 4(p_1^2 - 2p_2);$$

whence

$$t_0 + t_2 = -8, \quad t_1 + t_3 = 4;$$

and

$$t_3 = -3, \quad t_2 = 4.$$

Again, $t_6 = 0$; for Σ vanishes if $n - 2$ roots vanish. And we find t_4 and t_5 by taking the particular case when $n - 3$ roots vanish; for in this case

$$\Sigma \alpha_1^3 \alpha_2^2 \alpha_3 = \alpha_1 \alpha_2 \alpha_3 \Sigma \alpha_1^2 \alpha_2 = -p_3(-p_1 p_2 + 3p_3) = p_1 p_2 p_3 - 3p_3^2,$$

and therefore

$$t_4 = -3, \quad t_5 = 1;$$

whence, finally,

$$\Sigma \alpha_1^3 \alpha_2^2 \alpha_3 = -12p_6 + 7p_1 p_5 + 4p_4 p_2 - 3p_4 p_1^2 - 3p_3^2 + p_1 p_2 p_3.$$

147. **Seminvariants and Semicovariants.** — Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of

$$a_0 x^n + n a_1 x^{n-1} + \frac{n(n-1)}{1.2} a_2 x^{n-2} + \ldots + a_n = 0,$$

the general equation written with binomial coefficients (see Art. 35). We proceed to the discussion of an important class of functions of x which may be derived from a given symmetric function of the roots.

We called attention in Art. 36 to a class of homogeneous symmetric functions of the roots which contain the *differences only* of these quantities. Such functions may be called (for a reason which will appear in a subsequent chapter) *semi-invariants* or, as it is usually written, *seminvariants*. Being symmetric functions of the roots, they are expressible (when multiplied by a power of a_0) in a rational and integral form in terms of the coefficients.

We may use in like manner the term *semicovariants* to denote similar functions of the differences of the quantities $x, \alpha_1, \alpha_2, \ldots \alpha_n$, such that when they are arranged in powers of x the successive coefficients of x are expressible in a similar manner.

We proceed now to show how semicovariants may be generated, and then expanded in powers of x, when expressed either in terms of the roots or in terms of the coefficients.

From any relation such as

$$a_0^{\varpi}\phi(a_1, a_2, \ldots a_n) = F(a_0, a_1, a_2, \ldots a_n),$$

where ϕ is an integral function, of the order ϖ, and F the corresponding expression in terms of the coefficients, we may, by diminishing each of the roots by x, and consequently changing any coefficient a_r into U_r (see Art. 35), derive the following equation :—

$$a_0^{\varpi}\phi(a_1 - x,\ a_2 - x, \ldots a_n - x) = F(U_0, U_1, U_2, \ldots U_n), \quad (1)$$

thus obtaining two forms for a semicovariant, one expressed in terms of the roots, and the other in terms of the coefficients.

To expand these forms in powers of x, we have, for the first member of the equation, by Taylor's theorem,

$$\phi(a_1 - x, a_2 - x, \ldots a_n - x) = \phi_0 + x\delta\phi_0 + \frac{x^2}{1 \cdot 2}\delta^2\phi_0 + \ldots . \quad (2)$$

where $\phi_0 \equiv \phi(a_1, a_2, \ldots a_n)$,

and $-\delta \equiv \dfrac{d}{da_1} + \dfrac{d}{da_2} + \ldots + \dfrac{d}{da_n}$.

Again, omitting all powers of x higher than the first, the second member of the equation becomes

$$F(a_0,\ a_1 + a_0 x,\ a_2 + 2a_1 x, \ldots a_n + na_{n-1}x),$$

or, when expanded,

$$F_0 + xDF_0 + \&c.,$$

where

$$F_0 \equiv F(a_0, a_1, a_2, \ldots a_n),$$

and $D \equiv a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3} + \ldots + na_{n-1}\dfrac{d}{da_n}$.

Comparing the two expanded forms, we have

$$a_0^{\varpi}\delta\phi(a_1, a_2, \ldots a_n) = DF(a_0, a_1, \ldots a_n),$$

and consequently, by successive applications of the operators δ and D,

$$a_0^{\varpi}\delta^r\phi(a_1, a_2, \ldots a_n) = D^r F(a_0, a_1, \ldots a_n);$$

whence we infer from the expansion (2)

$$F(U_0, U_1, \ldots U_n) = F_0 + xDF_0 + \frac{x^2}{1.2}D^2F_0 + \&c. \ldots$$

By the aid, therefore, of the two operators—δ in terms of the roots, and D in terms of the coefficients—we can expand at pleasure either side of the equation (1) in powers of x. By means of the successive operations of δ we obtain a series of functions of the roots; and, by means of D, their equivalent values in terms of the coefficients.

The results now arrived at are equally true if the function ϕ involves the roots of two or more equations, F being the corresponding value in terms of the coefficients of these equations, and D and δ being replaced by the sums of the similar operators relative to each equation.

It is important to observe that when $\delta\phi_0$ vanishes identically, so also

$$\delta(\delta\phi_0) \text{ or } \delta^2\phi_0 = 0, \quad \delta^3\phi_0 = 0, \&c.,$$

and therefore x disappears in the expansion of the first member of equation (1). Now this can happen only when ϕ is a function of the differences of $a_1, a_2, \ldots a_n$; whence we conclude that if $F(a_0, a_1, \ldots a_n)$ is a seminvariant

$$DF(a_0, a_1, a_2, \ldots a_n) = 0.$$

This identical relation is generally sufficient to determine the numerical coefficients in a seminvariant when the order and weight are known, as the following examples will show. If there should be two or more seminvariants of the same order and weight, the operation of D will not supply equations enough to determine the assumed coefficients; and if no seminvariant exists of the required order and weight, the coefficients will all vanish.

Examples.

1. Determine for a cubic a seminvariant whose order and weight are both three.

Assume $$\phi = Aa_0^2 a_3 + Ba_0 a_1 a_2 + Ca_1^3,$$

these being the only three terms which satisfy the required conditions. It is evident from the form of D that the operation is performed by applying to the suffix of any coefficient a_r the same process as in ordinary differentiation is applied to the index. Thus $Da_r = ra_{r-1}$, and therefore

$$D\phi = (3A + B)\, a_0^2 a_2 + (2B + 3C)\, a_1^2 a_0 \equiv 0.$$

Hence

$$3A + B = 0, \quad \text{and} \quad 2B + 3C = 0\,;$$

and putting $A = 1$, we have $B = -3$, and $C = 2$;

whence, finally,

$$\phi = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \equiv G. \qquad \text{(See Art. 36.)}$$

For a quadratic no such seminvariant can be formed.

2. Determine for a cubic a seminvariant whose order is four, and weight six.

Assume

$$\phi = Aa_0^2 a_3^2 + Ba_0 a_2^3 + Ca_3 a_1^3 + Da_1^2 a_2^2 + Ea_0 a_1 a_2 a_3,$$

whence

$$D\phi = (6A + E)\, a_0^2 a_2 a_3 + (6B + 3E + 2D)\, a_0 a_1 a_2^2 + (3C + 4D)\, a_1^3 a_2 + (3C + 2E)\, a_0 a_1^2 a_3 \equiv 0.$$

Now let $A = 1$, whence $E = -6$; also $3C + 2E = 0$, giving $C = 4$; and $3C + 4D = 0$, giving $D = -3$; and from $6B + 3E + 2D = 0$, we have finally $B = 4$.

Hence

$$\phi = a_0^2 a_3^2 + 4a_0 a_2^3 + 4a_3 a_1^3 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3.$$

Compare Art. 42, where the value of ϕ is given in terms of the roots.

3. Prove directly that

$$\frac{d}{dx} F(U_0, U_1, U_2, \ldots U_n) = DF(U_0, U_1, U_2, \ldots U_n).$$

This follows readily from the equations

$$DU_r = rU_{r-1} = \frac{dU_r}{dx}\,; \; D\left\{U_p \,.\, U_q \,.\,.\right\} = \frac{d}{dx}\left\{U_p \,.\, U_q \,.\,.\right\}.$$

4. Expand $F(U_0, U_1, U_2, \ldots U_n)$ by Maclaurin's theorem; and hence prove

$$F(U_0, U_1, \ldots U_n) = F_0 + xDF_0 + \frac{x^2}{1\,.\,2} D^2F_0 + \&\text{c.},$$

where $$F_0 \equiv F(a_0, a_1, a_2, \ldots a_n).$$

148. **Theorem.**—We conclude this chapter with an important theorem connecting the leading coefficients of Sturm's functions and the sums of the powers of the roots of an equation $f(x) = 0$, viz., *The leading coefficients of Sturm's auxiliary functions* (i. e. $f'(x)$, *and the* $n - 1$ *remainders*) *differ by positive factors only from the following series of determinants* :—

$$s_0, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}, \quad \begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}, \text{ \&c.}$$

Using the bracket notation, we may write these determinants in the form s_0, $(s_0 s_2)$, $(s_0 s_2 s_4)$, &c., the last in the series being $(s_0 s_2 s_4 \ldots s_{2n-2})$.

Representing Sturm's remainders by R_2, R_3, . . . R_j, . . . R_n, and the successive quotients by Q_1, Q_2, Q_3, &c., we have (see Art. 89)

$$R_2 = Q_1 f'(x) - f(x),$$

$$R_3 = Q_2 R_2 - f'(x) = (Q_1 Q_2 - 1) f'(x) - Q_2 f(x),$$

$$R_4 = Q_3 R_3 - R_2 = (Q_1 Q_2 Q_3 - Q_1 - Q_3) f'(x) - (Q_2 Q_3 - 1) f(x), \text{ \&c.}$$

Proceeding in this manner, we observe that any remainder R_j can be expressed in the form

$$R_j = A_j f'(x) - B_j f(x). \qquad (1)$$

The degree of R_j is $n - j$; and since Q_1, Q_2, &c., are all of the first degree in x, it appears that the degrees of A_j and B_j are $j - 1$ and $j - 2$, respectively.

Assuming, therefore, for R_j and A_j the forms

$$R_j = r_0 + r_1 x + r_2 x^2 + \ldots + r_{n-j} x^{n-j},$$

$$A_j = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{j-1} x^{j-1};$$

and substituting in (1) any root a of the equation $f(x) = 0$, we have

$$\lambda_0 + \lambda_1 a + \lambda_2 a^2 + \ldots + \lambda_{j-1} a^{j-1} = \frac{r_0 + r_1 a + r_2 a^2 + \ldots + r_{n-j} a^{n-j}}{f'(a)}.$$

Multiplying by $a, a^2, \ldots a^{j-2}, a^{j-1}$, in succession; making similar substitutions of the other roots; and adding the equations thus derived, we obtain by aid of the relations of Ex. 4, p. 319, the following system of equations:—

$$\begin{aligned}
\lambda_0 s_0 + \lambda_1 s_1 + \ldots + \lambda_{j-2}\, s_{j-2} + \lambda_{j-1} s_{j-1} &= 0,\\
\lambda_0 s_1 + \lambda_1 s_2 + \ldots + \lambda_{j-2}\, s_{j-1} + \lambda_{j-1} s_j &= 0,\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot&\\
\lambda_0 s_{j-2} + \lambda_1 s_{j-1} + \ldots + \lambda_{j-2}\, s_{2j-4} + \lambda_{j-1} s_{2j-3} &= 0,\\
\lambda_0 s_{j-1} + \lambda_1 s_j + \ldots + \lambda_{j-2}\, s_{2j-3} + \lambda_{j-1} s_{2j-2} &= r_{n-j}.
\end{aligned}$$

From these equations we have, without difficulty,

$$r_{n-j} = \gamma_j \begin{vmatrix} s_0 & s_1 & \ldots & s_{j-1} \\ s_1 & s_2 & \ldots & s_j \\ \cdot & \cdot & \cdot & \cdot \\ s_{j-1} & s_j & \ldots & s_{2j-2} \end{vmatrix}, \quad A_j = \gamma_j \begin{vmatrix} s_0 & s_1 & \ldots & s_{j-2} & s_{j-1} \\ s_1 & s_2 & \ldots & s_{j-1} & s_j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{j-2} & s_{j-1} & \ldots & s_{2j-4} & s_{2j-3} \\ 1 & x & \ldots & x^{j-2} & x^{j-1} \end{vmatrix},$$

the value of γ_j being so far arbitrary. It appears therefore that the coefficient of the highest power of x in R_j differs by this multiplier only from the determinant $(s_0 s_2 s_4 \ldots . \; s_{2j-2})$. We proceed to show that the sign of γ_j is positive. For this purpose we make use of the following relation connecting the successive values of the functions R and A:—

$$A_{k+1} R_k - R_{k+1} A_k \equiv f(x). \tag{2}$$

To prove this; substituting for $R_{k+1} R_k$, R_{k-1} their values in terms of A and B in the relation $R_{k+1} = Q_k R_k - R_{k-1}$, we derive

$$A_{k+1} = Q_k A_k - A_{k-1}, \quad B_{k+1} = Q_k B_k - B_{k-1};$$

by aid of which we readily obtain the following relations connecting the successive functions:—

$$A_{k+1} B_k - A_k B_{k+1} = A_k B_{k-1} - A_{k-1} B_k = \ldots = A_1 B_0 - A_0 B_1 = -1,$$

$$A_{k+1} R_k - A_k R_{k+1} = A_k R_{k-1} - A_{k-1} R_k = \ldots = A_1 R_0 - A_0 R_1 = f(x),$$

in which $R_1 = f'(x)$, $R_0 = f(x)$.

Now, comparing the coefficients of the highest powers of x in (2); observing that x^n occurs only in $A_{k+1} R_k$, and making use of the determinant forms above obtained, we have

$$\gamma_{k+1} \left(s_0\, s_2\, s_4 \ldots s_{2k-2}\right) \gamma_k \left(s_0\, s_2\, s_4 \ldots s_{2k-2}\right) = 1,$$

or

$$\gamma_k\, \gamma_{k+1} = \left(s_0\, s_2\, s_4 \ldots s_{2k-2}\right)^{-2}.$$

Also, calculating the value of R_2 in the ordinary manner, we easily find

$$A_2 = \frac{1}{s_0^{\,2}} \begin{vmatrix} s_0 & s_1 \\ 1 & x \end{vmatrix};$$

whence it is seen that the value of γ_2 is $\dfrac{1}{s_0^{\,2}}$.

It follows, from the relation just established between any two successive values of γ, that $\gamma_3, \gamma_4, \ldots \gamma_j$, &c., are all positive squares; and therefore, finally, that r_{n-j}, the coefficient of the highest power of x in R_j, has the same sign as the determinant $\left(s_0 s_2 s_4 \ldots s_{2j-2}\right)$.

It should be noticed that there is only one way of expressing a function of x, of the degree $n-j$, in the form $Af'(x) - Bf(x)$, where A and B are of the degrees $j-1$ and $j-2$, respectively, and $f(x)$ of the degree n; for this function being in general of the degree $n+j-2$, in order that it may reduce to the degree $n-j$, the $2j-2$ highest terms must vanish, and this is exactly the number of undetermined quantities in A and B at our disposal, since it is the ratios only of the coefficients we are concerned with. Sturm's remainders may therefore be obtained in this way with an undetermined multiplier.

The functions R_j, A_j, and B_j are semicovariants of $f(x)$, as may be easily inferred by supposing $f(x)$ transformed by the substitution $z = a_0 x + a_1$ before these functions are calculated. The expression for A_j in terms of the differences of x and the roots will be found among the following examples; and the corresponding expressions for R_j and B_j will be given in a subsequent chapter, where we shall discuss Sylvester's theorem relating to Sturm's functions, in which the theorem of the present Article is included.

EXAMPLES.

1. Prove, by squaring the determinant of Example 10, Art. 122, the following relation between the roots α, β, γ, δ, of the biquadratic :—

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix} = (\beta-\gamma)^2(\alpha-\delta)^2(\gamma-\alpha)^2(\beta-\delta)^2(\alpha-\beta)^2(\gamma-\delta)^2.$$

The student will have no difficulty in writing down for an equation of any degree the corresponding determinant in terms of the sums of the powers of the roots which is equal to the product of the squares of the differences.

2. Prove, for the general equation,

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \Sigma(\alpha-\beta)^2.$$

This appears by squaring the array

$$\left.\begin{matrix} 1 & 1 & 1 & 1 & 1 & \ldots \\ \alpha & \beta & \gamma & \delta & \epsilon & \ldots \end{matrix}\right\} \quad \text{(See Art. 133.)}$$

3. Prove similarly, for the general equation,

$$\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2.$$

By the process of Art. 133, a series of relations of this kind can be established; and when the number of rows in the array becomes equal to the degree of the equation, the value of the determinant is the product of the squares of the differences, as in Ex. 1. When the number of rows exceeds the degree of the equation the value of the corresponding determinant vanishes. For example, the value of the determinant of Ex. 1 is zero for equations of the second and third degrees.

4. Prove, by means of the equations of Art. 140, that the sums of the powers can be expressed in terms of the coefficients, or *vice versa*, in the form of determinants, as follows :—

$$s_2 = \begin{vmatrix} p_1 & 1 \\ 2p_2 & p_1 \end{vmatrix}, \quad s_3 = -\begin{vmatrix} 1 & 1 & 0 \\ 2p_2 & p_1 & 1 \\ 3p_3 & p_2 & p_1 \end{vmatrix}, \quad s_4 = \begin{vmatrix} p_1 & 1 & 0 & 0 \\ 2p_2 & p_1 & 1 & 0 \\ 3p_3 & p_2 & p_1 & 1 \\ 4p_4 & p_3 & p_2 & p_1 \end{vmatrix}, \text{ \&c.}$$

$$2p_2 = \begin{vmatrix} s_1 & 1 \\ s_2 & s_1 \end{vmatrix}, \quad 6p_3 = -\begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 2 \\ s_3 & s_2 & s_1 \end{vmatrix}, \quad 24p_4 = \begin{vmatrix} s_1 & 1 & 0 & 0 \\ s_2 & s_1 & 2 & 0 \\ s_3 & s_2 & s_1 & 3 \\ s_4 & s_3 & s_2 & s_1 \end{vmatrix}, \text{ \&c.}$$

5. Resolve into factors the determinant

$$\begin{vmatrix} s_6 & s_5 & s_4 & s_3 & x^3 \\ s_5 & s_4 & s_3 & s_2 & x^2 \\ s_4 & s_3 & s_2 & s_1 & x \\ s_3 & s_2 & s_1 & s_0 & 1 \\ y^3 & y^2 & y & 1 & 0 \end{vmatrix},$$

where s_0, s_1, s_2, &c., are the sums of the powers of three quantities, α, β, γ.

This determinant is the product of the two determinants

$$\begin{vmatrix} \alpha^3 & \beta^3 & \gamma^3 & x^3 & 0 \\ \alpha^2 & \beta^2 & \gamma^2 & x^2 & 0 \\ \alpha & \beta & \gamma & x & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha^3 & \beta^3 & \gamma^3 & 0 & y^3 \\ \alpha^2 & \beta^2 & \gamma^2 & 0 & y^2 \\ \alpha & \beta & \gamma & 0 & y \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix};$$

and each of the latter can be resolved into simple factors.

6. Prove, for the general equation,

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ 1 & x & x^2 & x^3 \end{vmatrix} = \Sigma (\beta-\gamma)^2 (\gamma-\alpha)^2 (\alpha-\beta)^2 (x-\alpha)(x-\beta)(x-\gamma).$$

Multiplying the two arrays

$$\left.\begin{matrix} 1 & 1 & 1 & . & . \\ \alpha & \beta & \gamma & . & . \\ \alpha^2 & \beta^2 & \gamma^2 & . & . \end{matrix}\right\}, \quad \left.\begin{matrix} x-\alpha & x-\beta & x-\gamma & . & . \\ \alpha(x-\alpha) & \beta(x-\beta) & \gamma(x-\gamma) & . & . \\ \alpha^2(x-\alpha) & \beta^2(x-\beta) & \gamma^2(x-\gamma) & . & . \end{matrix}\right\},$$

we show that Σ is equal to

$$\begin{vmatrix} s_0x-s_1 & s_1x-s_2 & s_2x-s_3 \\ s_1x-s_2 & s_2x-s_3 & s_3x-s_4 \\ s_2x-s_3 & s_3x-s_4 & s_4x-s_5 \end{vmatrix},$$

which is easily transformed into the proposed determinant.

It appears in like manner in general that the determinant of similar form of order $p+1$ is equal to the corresponding symmetric function, each of whose terms contains p factors of the original equation multiplied by the product of the squared differences of the p roots involved therein.

7. Prove that the value of the quotient of A_j by γ_j (see Art. 148) may be written as a symmetric function involving x and the roots; e.g.,

$$\frac{A_4}{\gamma_4} = \Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(x-\alpha)(x-\beta)(x-\gamma).$$

8. Determine $\phi_1, \phi_2, \ldots \phi_j, \ldots \phi_p$ from the equations

$$\begin{aligned}
&\phi_1 + \phi_2 + \ldots + \phi_p = T_0,\\
&\phi_1\theta_1 + \phi_2\theta_2 + \ldots + \phi_p\theta_p = T_1,\\
&\phi_1\theta_1^2 + \phi_2\theta_2^2 + \ldots + \phi_p\theta_p^2 = T_2,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\phi_1\theta_1^{p-1} + \phi_2\theta_2^{p-1} + \ldots + \phi_p\theta_p^{p-1} = T_{p-1}.
\end{aligned}$$

Ans. ϕ_j is given as a function of the $(p-1)^{th}$ degree in θ_j by the equation

$$\begin{vmatrix}
1 & \theta_j & \theta_j^2 & \cdot & \cdot & \theta_j^{p-1} & \phi_j\\
s_0 & s_1 & s_2 & \cdot & \cdot & s_{p-1} & T_0\\
s_1 & s_2 & s_3 & \cdot & \cdot & s_p & T_1\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
s_{p-1} & s_p & s_{p+1} & \cdot & \cdot & s_{2p-2} & T_{p-1}
\end{vmatrix} = 0,$$

where $s_k = \theta_1^k + \theta_2^k + \theta_3^k + \ldots + \theta_p^k$.

9. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

calculate in terms of a_0, H, I, J the value of the symmetric function

$$a_0^6\Sigma(3\alpha-\beta-\gamma-\delta)^2(3\beta-\gamma-\delta-\alpha)^2(3\gamma-\delta-\alpha-\beta)^2.$$

Here
$$a_0^6\Sigma = 4^6\Sigma z_1^2 z_2^2 z_3^2,$$

where z_1, z_2, z_3, z_4 are the roots of the equation

$$z^4 + 6Hz^2 + 4Gz + a_0^2I - 3H^2 = 0. \qquad \text{(See Art. 37.)}$$

Hence, by Ex. 2, Art. 146,

Ans. $4^7\{-7H^3 + a_0^2HI - 4a_0^3J\}$.

10. Prove that

$$\Pi \equiv a_0^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2 = lI^3 + mJ^2,$$

where
$$m = -27l.$$

The weight of this function of the roots is 12, and the order 6.

We now make use of a proposition which will be proved subsequently, namely, that any even, rational, and integral symmetric function of the roots, of the order ϖ and weight κ, and involving the differences only of the roots, is, when multiplied by a_0^{ϖ}, a rational and *integral* function of a_0, H, I, J. (Compare Ex. 17, p. 126.)

Hence, expressing the function whose order is 6, and weight 12, in terms of a_0, H, I, J, it is easy to see from the table—

	Order.	Weight.
H	2	2
I	2	4
J	3	6

that H cannot enter, for the terms of the sixth order containing H, viz. H^3, H^2I, HI^2, have not the proper weight. Therefore Π must be of the form $lI^3 + mJ^2$, where l and m are numerical coefficients.

Now put a_3 and a_4 equal to zero, and Π will vanish, since in that case the quartic will have equal roots; hence, employing the reduced values of I and J,

$$0 = l(3a_2^2)^3 + m(-a_2^3)^2, \text{ and therefore } m = -27l.$$

In applying this method to obtain the values of symmetric functions, the rule to be followed in every case is—Retain those terms of weight κ whose order is not greater than ϖ, and make the whole homogeneous by multiplying terms whose order is less than ϖ by suitable powers of a_0.

11. Calculate the symmetric function of the roots of a biquadratic

$$\Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2.$$

Since the order of this symmetric function is four and its weight six, we may assume

$$a_0^4\Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = lHI + ma_0J. \tag{1}$$

The values of l and m may be found by putting $a_3 = 0$, $a_4 = 0$, as in the preceding example, and calculating the value of the reduced symmetric function (when $\gamma = 0$, $\delta = 0$) in terms of the coefficients of the quadratic equation

$$a_0x^2 + 4a_1x + 6a_2 = 0.$$

Identifying then this value with the reduced value of $lHI + ma_0J$, we obtain two simple equations to determine l and m. Or we may proceed as follows by taking two biquadratics whose roots are known, and calculating in each case the symmetric function by actually substituting the roots, and then comparing both sides of the equation when H, I, J are replaced by their values calculated from the numerical coefficients.

First we take the biquadratic equation $6x^4 - 6x^2 = 0$, whose roots are 0, 0, 1, -1, whence

$$\Sigma = 8, \quad H = -6, \quad I = 3, \quad J = 1.$$

Substituting in equation (1), we have

$$1728 = -3l + m.$$

Proceeding in the same way with the biquadratic equation

$$x^4 - 6x^2 + 5 = 0, \quad \text{whose roots are } \pm\sqrt{5},\ \pm 1,$$

we find

$$\Sigma = 768, \quad H = -1, \quad I = 8, \quad J = -4\,;$$

whence

$$-192 = 2l + m,$$

and

$$l = -2 \times 192, \quad m = 3 \times 192\,;$$

and finally,

$$a_0^4 \Sigma = 192\,(-2HI + 3a_0 J).$$

12. If $F(a_0, a_1, a_2, \ldots a_n)$ is a seminvariant of the equation $(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0$, prove that the same function of the sums of the powers of the roots, viz. $F(s_0, s_1, s_2, \ldots s_n)$, is also a seminvariant.

This follows by operating on the first function by D, and on the second by $-\delta$, and observing that $Da_r = ra_{r-1}$ and $-\delta s_r = rs_{r-1}$. We thus obtain results identical in form; and if one vanishes identically so must the other.

13. Calculate the determinant

$$\Delta \equiv \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

in terms of the coefficients of a quartic.

By the preceding example this determinant is a function of the differences of the roots; we may therefore remove the second term of the quartic before calculating it; and if the equation so transformed be

$$y^4 + P_2 y^2 + P_3 y + P_4 = 0,$$

$$\Delta = \begin{vmatrix} 4 & 0 & -2P_2 \\ 0 & -2P_2 & -3P_3 \\ -2P_2 & -3P_3 & 2P_2^2 - 4P_4 \end{vmatrix} = 4\,\{8P_2P_4 - 2P_2^3 - 9P_3^2\}\,;$$

but $$a_0^2 P_2 = 6H, \quad a_0^3 P_3 = 4G, \quad a_0^4 P_4 = a_0^2 I - 3H^2.$$

Substituting for P_2, P_3, P_4 these values, we have

$$a_0^4 \Delta = 192\,(-2HI + 3a_0 J):$$

the same result as in the preceding example (cf. Ex. 3, p. 336).

14. If α, β, γ, δ be the roots of the equation

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0,$$

express H_s, I_s, J_s, G_s of the equation

$$s_0x^4 + 4s_1x^3 + 6s_2x^2 + 4s_3x + s_4 \equiv \Sigma(x+a)^4 = 0$$

in terms of H, I, J, G.

$$\textit{Ans.}\ \frac{H_s}{s_0^2} = -3\frac{H}{a_0^2},\quad \frac{I_s}{s_0^2} = \frac{48H^2 - a_0^2 I}{a_0^4},\quad \frac{G_s}{s_0^3} = -3\frac{G}{a_0^3};$$

and by the aid of the relations

$$G^2 + 4H^3 \equiv a_0^2(HI - a_0J),\quad G_s^2 + 4H_s^3 \equiv s_0^2(H_sI_s - s_0J_s),$$

$$J_s = \frac{192}{a_0^4}(3a_0J - 2HI).$$

15. When p is even, prove that

$$\Sigma(\alpha_1 - \alpha_2)^p = s_0 s_p - ps_1 s_{p-1} + \tfrac{1}{2}p(p-1)s_2 s_{p-2} - \&c.$$

Since

$$\Sigma(x-\alpha)^p = nx^p - ps_1x^{p-1} + \frac{p\,.\,p-1}{2}s_2x^{p-2} - \&c.\ldots - ps_{p-1}x + s_p,$$

changing x into $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, in succession, and adding the results on both sides of the equations thus obtained, we find

$$2\Sigma(\alpha_1 - \alpha_2)^p = s_0s_p - ps_1s_{p-1} + \frac{p\,.\,p-1}{1\,.\,2}s_2s_{p-2} - \ldots - ps_1s_{p-1} + s_0s_p,$$

where all the terms on the right side of this equation are repeated except the middle term. Thus

$$\Sigma(\alpha_1 - \alpha_2)^4 = s_0s_4 - 4s_1s_3 + 3s_2^2,$$

$$\Sigma(\alpha_1 - \alpha_2)^6 = s_0s_6 - 6s_1s_5 + 15s_2s_4 - 10s_3^2,\ \&c.$$

16. Form the equation whose roots are $\phi'(\alpha), \phi'(\beta), \phi'(\gamma), \phi'(\delta)$, where $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$\phi(x) \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

$$\textit{Ans.}\ \phi'^4 + \frac{32G}{a_0^3}\phi'^3 + \frac{96(2HI - 3a_0J)}{a_0^4}\phi'^2 + \frac{256(I^3 - 27J^2)}{a_0^6} = 0.$$

17. If $$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)^4,$$

when expanded, becomes

$$K_0x^4 + 4K_1x^3 + 6K_2x^2 + 4K_3x + K_4;$$

prove that

$$\frac{K_0\alpha\beta\gamma + K_1(\beta\gamma + \gamma\alpha + \alpha\beta) + K_2(\alpha + \beta + \gamma) + K_3}{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)} = \frac{\pm 16\sqrt{\Delta}}{a_0^3},$$

where

$$\Delta = I^3 - 27J^2.$$

18. Prove that

$$a_0^4\Sigma(\beta+\gamma-\alpha-\delta)^2(\beta-\gamma)^2(\alpha-\delta)^2 = 192(3a_0J - 2HI).$$

19. Prove that

$$a_0^6 \Sigma (\beta + \gamma - \alpha - \delta)^4 (\beta - \gamma)^2 (\alpha - \delta)^2 = 512 (a_0^2 I^2 - 36 a_0 HJ + 12 H^2 I).$$

20. The quotient of a simple alternant (one, namely, in which each element is a single power) by the difference-product (see Ex. 31, p. 303) can be expressed as a determinant whose elements are the sums of the homogeneous products of the quantities involved.

We take a determinant of the third order, and propose to prove

$$\begin{vmatrix} \alpha^p & \alpha^q & \alpha^r \\ \beta^p & \beta^q & \beta^r \\ \gamma^p & \gamma^q & \gamma^r \end{vmatrix} \equiv \begin{vmatrix} \Pi_p & \Pi_q & \Pi_r \\ \Pi_{p-1} & \Pi_{q-1} & \Pi_{r-1} \\ \Pi_{p-2} & \Pi_{q-2} & \Pi_{r-2} \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix},$$

where Π_p, Π_q, &c., are the sums of the homogeneous products of α, β, γ, as defined in Ex. 6, Art. 143. The method employed is perfectly general. Take the following identity, which is easily proved:—

$$\begin{vmatrix} \dfrac{x}{x-\alpha} & \dfrac{y}{y-\alpha} & \dfrac{z}{z-\alpha} \\[2ex] \dfrac{x}{x-\beta} & \dfrac{y}{y-\beta} & \dfrac{z}{z-\beta} \\[2ex] \dfrac{x}{x-\gamma} & \dfrac{y}{y-\gamma} & \dfrac{z}{z-\gamma} \end{vmatrix} \equiv \frac{\begin{vmatrix} x^3 & y^3 & z^3 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}}{(x-\alpha)(x-\beta)(x-\gamma)(y-\alpha)(y-\beta)(y-\gamma)(z-\alpha)(z-\beta)(z-\gamma)};$$

write $(x-\alpha)(x-\beta)(x-\gamma)$ as a divisor under each of the elements of the first column on the right-hand side, $(y-\alpha)(y-\beta)(y-\gamma)$ under those of the second, and $(z-\alpha)(z-\beta)(z-\gamma)$ under those of the third, and substitute from the following and similar equations (see Ex. 6, Art. 143):—

$$\frac{x}{x-\alpha} = 1 + \alpha x' + \alpha^2 x'^2 + \ldots + \alpha^p x'^p + \ldots,$$

$$\frac{x^3}{(x-\alpha)(x-\beta)(x-\gamma)} = 1 + \Pi_1 x' + \Pi_2 x'^2 + \ldots + \Pi_p x'^p + \&c.$$

where $$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad z' = \frac{1}{z}.$$

The identity written above becomes then

$$\begin{vmatrix} 1 + \alpha x' + \ldots + \alpha^p x'^p + \ldots \\ 1 + \beta x' + \ldots + \beta^p x'^p + \ldots \\ 1 + \gamma x' + \ldots + \gamma^p x'^p + \ldots \end{vmatrix} \equiv \begin{vmatrix} 1 + \Pi_1 x' + \Pi_2 x'^2 + \ldots + \Pi_p x'^p + \ldots \\ x' + \Pi_1 x'^2 + \ldots + \Pi_{p-1} x'^p + \ldots \\ x'^2 + \ldots + \Pi_{p-2} x'^p + \ldots \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix},$$

where the second and third columns of the determinants here written can be supplied by replacing x' by y' and z', respectively. Comparing coefficients of $x'^p y'^q z'^r$

on both sides, we have the required result. It should be noticed that when the difference-product determinant is written in the form used above (viz. with ascending powers in the order of the columns) the sign to be attached to the product is always positive, since the product of the two determinants, containing the term $\Pi_p \Pi_{q-1} \Pi_{r-2} \beta\gamma^2$, must contain the term $\alpha^p \beta^q \gamma^r$. Note also, in applying this calculation to particular examples, that $\Pi_o = 1$, and $\Pi_j = 0$ when j is negative.

21. Prove, by the preceding example,

$$\begin{vmatrix} 1 & \alpha^2 & \alpha^5 \\ 1 & \beta^2 & \beta^5 \\ 1 & \gamma^2 & \gamma^5 \end{vmatrix} = \begin{vmatrix} \Pi_0 & \Pi_2 & \Pi_5 \\ 0 & \Pi_1 & \Pi_4 \\ 0 & \Pi_0 & \Pi_3 \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

The quotient, therefore, of the given determinant by the difference-product is $\Pi_1 \Pi_3 - \Pi_4$, which may be shown to be equal to $\Sigma\alpha^3\beta + \Sigma\alpha^2\beta^2 + 2\Sigma\alpha^2\beta\gamma$.

22. Prove, by the method of Ex. 20,

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdot & \cdot & \alpha_1^{n-2} & \alpha_1^m \\ 1 & \alpha_2 & \alpha_2^2 & \cdot & \cdot & \alpha_2^{n-2} & \alpha_2^m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \alpha_n & \alpha_n^2 & \cdot & \cdot & \alpha_n^{n-2} & \alpha_n^m \end{vmatrix} = \Pi_{m-n+1} \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdot & \cdot & \cdot & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdot & \cdot & \cdot & \alpha_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \alpha_n & \alpha_n^2 & \cdot & \cdot & \cdot & \alpha_n^{n-1} \end{vmatrix},$$

where $m =$ or $> n$.

This result may be derived directly from Ex. 6, Art. 143.

23. Determine a seminvariant of a quintic whose order is three and weight five.

Ans. $a_0^2 a_5 - 5a_0 a_1 a_4 + 2a_0 a_2 a_3 - 6a_1 a_2^2 + 8a_1^2 a_3$.

24. Determine a seminvariant of a sextic whose order is three and weight eight.

Ans. $a_0 a_2 a_6 - 3a_0 a_3 a_5 + 2a_0 a_4^2 - a_1^2 a_6 + 3a_1 a_2 a_5 - a_1 a_3 a_4 - 3a_2^2 a_4 + 2a_2 a_3^2$.

25. Prove that any seminvariant of the equation $(a_0, a_1, \ldots a_r)(x, 1)^r = 0$ is also a seminvariant of the equation $(a_0, a_1, \ldots a_r, \ldots a_n)(x, 1)^n = 0$, n being greater than r.

CHAPTER XIV.

ELIMINATION.

149. **Definitions.**—Being given a system of n equations, homogeneous between n variables, or non-homogeneous between $n-1$ variables, if we combine these equations in such a manner as to eliminate the variables, and obtain an equation $R = 0$, containing only the coefficients of the equations; the quantity R is, when expressed in a rational and integral form, called the *Resultant* or *Eliminant*.

In what follows we shall be concerned chiefly with two equations involving one unknown quantity x only. In this case the equation $R = 0$ asserts that the two equations are consistent; that is, they are both satisfied by a common value of x. We now proceed to show how the elimination may be performed so as to obtain the quantity R, illustrating the different methods by simple examples. It is proper to observe that the value of R arrived at by some processes of elimination may contain a redundant factor. The method of elimination by symmetric functions leads to a value of R free from any such factor; and we refer, therefore, to the conclusion of the discussion in the next Article for the precise definition of the *Resultant*.

Let it be required to eliminate x between the equations

$$ax^2 + 2bx + c = 0, \quad a'x^2 + 2b'x + c' = 0.$$

Solving these equations, and equating the value of x so obtained, the result of elimination appears in the irrational form

$$-\frac{b}{a} + \frac{\sqrt{b^2 - ac}}{a} = -\frac{b'}{a'} + \frac{\sqrt{b'^2 - a'c'}}{a'}.$$

Multiplying by aa' we obtain

$$ab' - a'b = a\sqrt{b'^2 - a'c'} - a'\sqrt{b^2 - ac}.$$

Squaring both sides, and dividing by the redundant factor $a\,a'$, and then squaring again, we find

$$R = 4\,(ac - b^2)\,(a'c' - b'^2) - (ac' + a'c - 2bb')^2.$$

This method of forming the resultant is very limited in application, as it is not, in general, possible to express by an algebraic formula a root of an equation higher than the fourth degree. Other methods have consequently been devised for determining the resultant without first solving the equations. We now proceed to explain the method of elimination by symmetric functions of the roots of the equations.

150. **Elimination by Symmetric Functions.**—Let two algebraic equations of the m^{th} and n^{th} degrees be

$$\phi\,(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0,$$

$$\psi\,(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0;$$

and let it be required to find the condition that these equations should have a common root. For this purpose let the roots of the equation $\phi\,(x) = 0$ be $\alpha_1, \alpha_2, \ldots \alpha_m$. If the given equations have a common root it is *necessary* and *sufficient* that one of the quantities

$$\psi\,(\alpha_1), \quad \psi\,(\alpha_2), \ldots, \quad \psi\,(\alpha_m)$$

should be zero, or, in other words, that the product

$$\psi\,(\alpha_1)\,\psi\,(\alpha_2)\,\psi\,(\alpha_3)\ldots\psi\,(\alpha_m)$$

should vanish. If, now, we transform this product into a rational and integral function of the coefficients, which is always possible as it is a symmetric function of the roots of the equation $\phi\,(x) = 0$, we shall have the resultant required. Further, if $\beta_1, \beta_2, \ldots \beta_n$ be the roots of the equation $\psi\,(x) = 0$, we have

$$\psi\,(\alpha_1) = b_0\,(\alpha_1 - \beta_1)\,(\alpha_1 - \beta_2)\ldots(\alpha_1 - \beta_n),$$

$$\psi\,(\alpha_2) = b_0\,(\alpha_2 - \beta_1)\,(\alpha_2 - \beta_2)\ldots(\alpha_2 - \beta_n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\psi\,(\alpha_m) = b_0\,(\alpha_m - \beta_1)\,(\alpha_m - \beta_2)\ldots(\alpha_m - \beta_n).$$

If we change the signs of the $m\,n$ factors, and multiply these equations, taking together the factors which are situated in the same column, we find

$$a_0^n\,\psi(a_1)\,\psi(a_2)\,.\,.\,.\,\psi(a_m) = (-1)^{mn}\,b_0^m\phi(\beta_1)\,\phi(\beta_2)\,.\,.\,.\,\phi(\beta_n).$$

We may therefore take

$$R = (-1)^{mn}\,b_0^m\phi(\beta_1)\,\phi(\beta_2)\,.\,.\,.\,\phi(\beta_n) = a_0^n\,\psi(a_1)\,\psi(a_2)\,.\,.\,.\,\psi(a_m), \quad (1)$$

for both these values of R are integral functions of the coefficients of $\phi(x)$ and $\psi(x)$, which vanish only when $\phi(x)$ and $\psi(x)$ have a common factor, and which become identical when they are expressed in terms of the coefficients.

151. **Properties of the Resultant.**—(1). *The order of the resultant of two equations in the coefficients is equal to the sum of the degrees of the equations, the coefficients of the first equation entering R in the degree of the second, and the coefficients of the second entering in the degree of the first.*

This appears by reviewing the two forms of R in (1), Art. 150; for in the first form $a_0, a_1, \ldots a_m$ enter in the n^{th} degree, and in the second form $b_0, b_1, \ldots b_n$ enter in the m^{th} degree. Also it may be seen that two terms, one selected from each form, are $(-1)^{mn}\,b_0^m\,a_m^n$ and $a_0^n\,b_n^m$.

(2). *If the roots of both equations be multiplied by the same quantity ρ, the resultant is multiplied by ρ^{mn}.*

This is evident, since any one of the $m\,n$ factors of the form $a_p - \beta_q$ becomes $\rho\,(a_p - \beta_q)$, and therefore ρ^{mn} divides the resultant. From this we may conclude that *the weight of the resultant is mn*, in which form this proposition is often stated.

(3). *If the roots of both equations be increased by the same quantity, the resultant of the equations so transformed is equal to the resultant of the original equations.*

For we have

$$\pm\,R = a_0^n\,b_0^m\,\Pi\,(a_p - \beta_q),$$

where Π signifies the continued product of the mn terms of the form $a_p - \beta_q$; and this is unaltered when a_p and β_q receive the same increment.

(4). *If the roots be changed into their reciprocals, the value of R obtained from the transformed equations remains unaltered, except in sign when mn is an odd number.*

Making this transformation in

$$R = a_0^n b_0^m \Pi (\alpha_p - \beta_q),$$

we have

$$R' = a_m^n b_n^m (-1)^{mn} \frac{\Pi (\alpha_p - \beta_q)}{(\alpha_1\alpha_2 \ldots \alpha_m)^n (\beta_1\beta_2 \ldots \beta_n)^m};$$

but

$$\alpha_1\alpha_2 \ldots \alpha_m = (-1)^m \frac{a_m}{a_0}, \quad \beta_1\beta_2 \ldots \beta_n = (-1)^n \frac{b_n}{b_0};$$

substituting, we obtain

$$R' = a_0^n b_0^m (-1)^{mn} \Pi (\alpha_p - \beta_q) = (-1)^{mn} R.$$

From this it follows that in the resultant of two equations the coefficients with complementary suffixes of both equations, e.g. a_0, a_m; a_1, a_{m-1}, &c., may be all interchanged without altering the value of the resultant.

(5). *If both equations be transformed by homographic transformation, that is, by substituting for x*

$$\frac{\lambda x + \mu}{\lambda' x + \mu'},$$

and each simple factor multiplied by $\lambda' x + \mu'$, *to render the new equations integral; then the new resultant* $R' = (\lambda\mu' - \lambda'\mu)^{mn} R$.

To prove this, we have

$$\phi (x) = a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m),$$

$$\psi (x) = b_0 (x - \beta_1)(x - \beta_2) \ldots (x - \beta_n);$$

also

$$x - \alpha_r \text{ becomes } (\lambda - \lambda'\alpha_r)\left(x - \frac{\mu'\alpha_r - \mu}{\lambda - \lambda'\alpha_r}\right),$$

$$x - \beta_r \quad ,, \quad (\lambda - \lambda'\beta_r)\left(x - \frac{\mu'\beta_r - \mu}{\lambda - \lambda'\beta_r}\right).$$

Multiplying together all the factors of each equation,

$$a_0 \text{ becomes } a_0 (\lambda - \lambda'\alpha_1)(\lambda - \lambda'\alpha_2) \ldots (\lambda - \lambda'\alpha_m),$$

$$b_0 \quad ,, \quad b_0 (\lambda - \lambda'\beta_1)(\lambda - \lambda'\beta_2) \ldots (\lambda - \lambda'\beta_n).$$

Also, since a_r, β_r are transformed into $\frac{\mu' a_r - \mu}{\lambda - \lambda' a_r}$, $\frac{\mu'\beta_r - \mu}{\lambda - \lambda'\beta_r}$,

$$a_r - \beta_r \text{ becomes } \frac{(\lambda\mu' - \lambda'\mu)(a_r - \beta_r)}{(\lambda - \lambda' a_r)(\lambda - \lambda'\beta_r)};$$

whence

$$a_0{}^n b_0{}^m \Pi (a_r - \beta_r) \text{ becomes } a_0{}^n b_0{}^m (\lambda\mu' - \lambda'\mu)^{mn} \Pi (a_r - \beta_r),$$

that is, the resultant calculated from the new forms of $\phi(x)$ and $\psi(x)$ is

$$(\lambda\mu' - \lambda'\mu)^{mn} R.$$

This proposition includes the three foregoing; and they are collectively equivalent to the present proposition.

152. **Euler's Method of Elimination.**—When two equations $\phi(x) = 0$, and $\psi(x) = 0$, of the m^{th} and n^{th} degrees respectively, have any common root θ, we may assume

$$\phi(x) \equiv (x - \theta)\,\phi_1(x),$$

$$\psi(x) \equiv (x - \theta)\,\psi_1(x),$$

where

$$\phi_1(x) \equiv p_1 x^{m-1} + p_2 x^{m-2} + \ldots + p_m,$$

$$\psi_1(x) \equiv q_1 x^{n-1} + q_2 x^{n-2} + \ldots + q_n,$$

the coefficients being undetermined quantities depending on θ. Whence we have

$$\phi(x)\,\psi_1(x) \equiv \psi(x)\,\phi_1(x),$$

an identical equation of the $(m+n-1)^{th}$ degree. Now, equating the coefficients of the different powers of x on both sides of the equation, we have $m+n$ homogeneous equations of the first degree in the $m+n$ quantities $p_1, p_2, \ldots p_m, q_1, q_2, \ldots q_n$; and eliminating these quantities by the method of Art. 135, we obtain the resultant of the two given equations in the form of a determinant.

EXAMPLE.

Suppose the two equations

$$ax^2 + bx + c = 0, \quad a_1 x^2 + b_1 x + c_1 = 0$$

to have a common root. We have identically

$$(q_1 x + q_2)(ax^2 + bx + c) \equiv (p_1 x + p_2)(a_1 x^2 + b_1 x + c_1),$$

or

$$(q_1 a - p_1 a_1) x^3 + (q_1 b + q_2 a - p_1 b_1 - p_2 a_1) x^2 + (q_1 c + q_2 b - p_1 c_1 - p_2 b_1) x + q_2 c - p_2 c_1 \equiv 0.$$

Equating to zero all the coefficients of this equation, we have the four homogeneous equations

$$\begin{aligned} q_1 a \qquad\;\; - p_1 a_1 \qquad\;\; &= 0, \\ q_1 b + q_2 a - p_1 b_1 - p_2 a_1 &= 0, \\ q_1 c + q_2 b - p_1 c_1 - p_2 b_1 &= 0, \\ q_2 c \qquad\;\; - p_2 c_1 &= 0; \end{aligned}$$

and eliminating p_1, p_2, q_1, q_2, we obtain the condition for a common root in the form

$$\begin{vmatrix} a & 0 & a_1 & 0 \\ b & a & b_1 & a_1 \\ c & b & c_1 & b_1 \\ 0 & c & 0 & c_1 \end{vmatrix} = 0.$$

The student can easily verify that this result is the same as that of Art. 149.

153. **Sylvester's Dialytic Method of Elimination.**—This method leads to the same determinants for resultants as the method of Euler just explained; but it has an advantage over Euler's method in point of generality, since it can often be applied to form the resultant of equations involving several variables.

Suppose we require the resultant of the two equations

$$\phi(x) \equiv a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m = 0,$$
$$\psi(x) \equiv b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \ldots + b_n = 0,$$

we multiply the first by the successive powers of x,

$$x^{n-1}, x^{n-2}, \ldots x^2, x, x^0;$$

and the second by $x^{m-1}, x^{m-2}, \ldots x^2, x, x^0$,

thus obtaining $m + n$ equations, the highest power of x being $m + n - 1$. We have, consequently, equations enough from which to eliminate

$$x^{m+n-1}, x^{m+n-2}, \ldots x^2, x.$$

considered as distinct variables.

EXAMPLES.

1. Find the resultant R of two quadratic equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0.$$

We have

$$\begin{aligned} ax^3 + bx^2 + cx \quad &= 0, \\ ax^2 + bx + c &= 0, \\ a_1x^3 + b_1x^2 + c_1x \quad &= 0, \\ a_1x^2 + b_1x + c_1 &= 0; \end{aligned}$$

from which, eliminating x^3, x^2, x, we get the same determinant as in the preceding Article, columns now replacing rows:

$$R \equiv \begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \end{vmatrix}.$$

2. Form the resultant of the two equations

$$U \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = 0,$$

$$V \equiv b_0 + b_1x + b_2x^2 + b_3x^3 = 0.$$

Proceeding as before, we easily find

$$R \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

It will be observed that R contains the coefficients of U in the 3rd degree, and those of V in the 4th degree; also $a_0{}^3b_3{}^4$ is a term in R (see (1), Art. 151).

154. **Bezout's Method of Elimination.**—The general method will be most easily understood by applying it in the first instance to particular cases. We proceed to this application—(1) when the equations are of the same degree, and (2) when they are of different degrees.

(1). Let us take the two cubic equations

$$ax^3 + bx^2 + cx + d = 0, \quad a_1x^3 + b_1x^2 + c_1x + d_1 = 0.$$

Multiplying these two equations successively by

$$\begin{array}{ccc} a_1 & \text{and} & a, \\ a_1x + b_1 & ,, & ax + b, \\ a_1x^2 + b_1x + c_1 & ,, & ax^2 + bx + c, \end{array}$$

and subtracting each time the products so formed, we find the three following equations:—

$$\begin{aligned} &(ab_1)\,x^2 + (ac_1)\,x + (ad_1) = 0, \\ &(ac_1)\,x^2 + \{(ad_1) + (bc_1)\}\,x + (bd_1) = 0, \\ &(ad_1)\,x^2 + (bd_1)\,x + (cd_1) = 0. \end{aligned}$$

By eliminating from these equations x^2, x, as distinct variables, the resultant is obtained in the form of a symmetrical determinant as follows:—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) \\ (ac_1) & (ad_1) + (bc_1) & (bd_1) \\ (ad_1) & (bd_1) & (cd_1) \end{vmatrix}.$$

To render the law of formation of the resultant more apparent, the following mode of procedure is given.

Let the two equations be biquadratics, as follows:—

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 = 0;$$

whence, following Cauchy's mode of presenting Bezout's method, we have the system of equations

$$\frac{a}{a_1} = \frac{bx^3 + cx^2 + dx + e}{b_1x^3 + c_1x^2 + d_1x + e_1},$$

$$\frac{ax + b}{a_1x + b_1} = \frac{cx^2 + dx + e}{c_1x^2 + d_1x + e_1},$$

$$\frac{ax^2 + bx + c}{a_1x^2 + b_1x + c_1} = \frac{dx + e}{d_1x + e_1},$$

$$\frac{ax^3 + bx^2 + cx + d}{a_1x^3 + b_1x^2 + c_1x + d_1} = \frac{e}{e_1},$$

which, when rendered integral, lead, on the elimination of x^3, x^2, x, to the following form for the resultant :—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) \\ (ac_1) & (ad_1) + (bc_1) & (ae_1) + (bd_1) & (be_1) \\ (ad_1) & (ae_1) + (bd_1) & (be_1) + (cd_1) & (ce_1) \\ (ae_1) & (be_1) & (ce_1) & (de_1) \end{vmatrix}.$$

If, now, we consider the two symmetrical determinants

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) \\ (ac_1) & (ad_1) & (ae_1) & (be_1) \\ (ad_1) & (ae_1) & (be_1) & (ce_1) \\ (ae_1) & (be_1) & (ce_1) & (de_1) \end{vmatrix}, \quad \begin{vmatrix} (bc_1) & (bd_1) \\ (bd_1) & (cd_1) \end{vmatrix},$$

the formation of which is at once apparent, we observe that R is obtained by adding the constituents of the second to the four central constituents of the first.

Similarly in the case of the two equations of the fifth degree

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

$$a_1x^5 + b_1x^4 + c_1x^3 + d_1x^2 + e_1x + f_1 = 0,$$

the resultant is obtained from the three following determinants :—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) & (af_1) \\ (ac_1) & (ad_1) & (ae_1) & (af_1) & (bf_1) \\ (ad_1) & (ae_1) & (af_1) & (bf_1) & (cf_1) \\ (ae_1) & (af_1) & (bf_1) & (cf_1) & (df_1) \\ (af_1) & (bf_1) & (cf_1) & (df_1) & (ef_1) \end{vmatrix}, \quad \begin{vmatrix} (bc_1) & (bd_1) & (be_1) \\ (bd_1) & (be_1) & (ce_1) \\ (be_1) & (ce_1) & (de_1) \end{vmatrix}, \quad (cd_1),$$

by adding the constituents of the second to the nine central constituents of the first, and then adding the third to the central constituent of the determinant so formed. The student will have no difficulty in applying a similar process of superposition to the formation of the determinant in general.

(2). We take now the case of two equations of different dimensions, for example,

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a_1x^2 + b_1x + c_1 = 0.$$

Multiplying these equations successively by

$$\begin{array}{ccc} a_1 & \text{and} & ax^2, \\ a_1x + b_1 & ,, & (ax+b)\,x^2, \end{array}$$

and subtracting each time the products so formed, we find the two following equations:—

$$(ab_1)\,x^3 + (ac_1)\,x^2 - da_1x - ea_1 = 0,$$

$$(ac_1)\,x^3 + \{(bc_1) - da_1\}\,x^2 - \{db_1 + ea_1\}\,x - eb_1 = 0.$$

If, now, we join to these the two equations

$$a_1x^3 + b_1x^2 + c_1x = 0,$$

$$a_1x^2 + b_1x + c_1 = 0,$$

we shall have four equations by means of which x^3, x^2, x can be eliminated; whence we obtain the resultant in the form of a determinant as follows:—

$$\begin{vmatrix} (ab_1) & (ac_1) & da_1 & ea_1 \\ (ac_1) & (bc_1) - da_1 & db_1 + ea_1 & eb_1 \\ a_1 & b_1 & -c_1 & 0 \\ 0 & a_1 & -b_1 & -c_1 \end{vmatrix}.$$

This determinant involves the coefficients of the first equation in the second degree, and the coefficients of the second equation in the fourth degree, as it should do; whence no extraneous factor enters this form of the resultant.

We now proceed to the general case of two equations of the m^{th} and n^{th} degrees.

Let the equations be

$$\phi\,(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0,$$

$$\psi\,(x) = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0,$$

where $m > n$; and let the second equation be multiplied by x^{m-n}. We have then

$$b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \ldots + b_nx^{m-n} = 0,$$

an equation of the same degree as the first. This equation has, however, in addition to the n roots of $\psi(x) = 0$, $m - n$ zero roots; so that we must be on our guard lest the factor $a_m{}^{m-n}$ (i.e. the result of substituting these roots in $\phi(x)$) enter the form of the resultant obtained. From these two equations we derive, as in the above case—(1), the following n equations:—

$$\frac{a_0}{b_0} = \frac{a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m}{b_1x^{m-1} + b_2x^{m-2} + \ldots + b_nx^{m-n}},$$

$$\frac{a_0x + a_1}{b_0x + b_1} = \frac{a_2x^{m-2} + a_3x^{m-3} + \ldots + a_m}{b_2x^{m-2} + b_3x^{m-3} + \ldots + b_nx^{m-n}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{a_0x^{n-1} + a_1x^{n-2} + \ldots + a_{n-1}}{b_0x^{n-1} + b_1x^{n-2} + \ldots + b_{n-1}} = \frac{a_nx^{m-n} + a_{n-1}x^{m-n-1} + \ldots + a_m}{b_nx^{m-n}},$$

which, when rendered integral, are all of the $(m-1)^{th}$ degree; whence, eliminating $x^{m-1}, x^{m-2}, \ldots x$ as independent quantities between these n and the $m - n$ equations

$$\begin{aligned} b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \ldots &= 0, \\ b_0x^{m-2} + b_1x^{m-3} + \ldots &= 0, \\ \cdots\cdots\cdots & \\ b_0x^n + b_1x^{n-1} + \ldots + b_n &= 0, \end{aligned}$$

we obtain the resultant in the form of a determinant of the m^{th} order, the coefficients of the first equation entering in the degree n, and the coefficients of the second equation entering in the degree m; whence it appears that no extraneous factor can enter; and that the resultant as obtained by this method has not been affected by the introduction of the zero roots.

If R be the resultant of two equations, $\phi(x) = 0$, $\psi(x) = 0$, whose degrees are both equal to m, the resultant R' of the system

$$\lambda\phi(x) + \mu\psi(x) = 0, \quad \lambda'\phi(x) + \mu'\psi(x) = 0$$

is
$$(\lambda\mu' - \lambda'\mu)^m R;$$

for each of the minors $(a_r b_s)$, which in Bezout's method constitute the determinant form of R, becomes in this case

$$\begin{vmatrix} \lambda a_r + \mu b_r, & \lambda' a_r + \mu' b_r \\ \lambda a_s + \mu b_s, & \lambda' a_s + \mu' b_s \end{vmatrix} = (\lambda\mu' - \lambda'\mu)(a_r b_s);$$

whence $R' = (\lambda\mu' - \lambda'\mu)^m R$, since R is a determinant of order m.

155. **Other Methods of Elimination**.—We conclude the subject of Elimination with an account of a method which is often employed, but which has the disadvantage, when applied to equations of higher degree than the second, of giving the resultant multiplied by extraneous factors. The process about to be explained is virtually equivalent to that usually described as the method of the greatest common measure.

In forming by this method the resultant of the two quadratic equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0,$$

we multiply these equations successively by

$$a_1 \text{ and } a, \qquad c_1 \text{ and } c,$$

and subtract the products so formed. We thus find the two linear equations

$$(ab_1)x + (ac_1) = 0,$$
$$(ac_1)x + (bc_1) = 0;$$

from which, eliminating x, we have

$$(ac_1)^2 - (ab_1)(bc_1) = 0.$$

As the degree of this expression is four, and its weight four, it can contain no extraneous factor, and is a correct form for the resultant.

To form by the same process the resultant of the cubic equations

$$ax^3 + bx^2 + cx + d = 0, \quad a_1x^3 + b_1x^2 + c_1x + d_1 = 0,$$

we multiply these equations successively by a_1 and a, d_1 and d, and subtract each time the products so formed. We have then

$$\begin{aligned}(ab_1)\,x^2 + (ac_1)\,x + (ad_1) &= 0,\\ (ad_1)\,x^2 + (bd_1)\,x + (cd_1) &= 0.\end{aligned} \qquad (1)$$

Now, eliminating x between these two quadratics by means of the formula above obtained, we find for their resultant

$$\begin{vmatrix} (ab_1) & (ad_1) \\ (ad_1) & (cd_1) \end{vmatrix}^2 - \begin{vmatrix} (ab_1) & (ac_1) \\ (ad_1) & (bd_1) \end{vmatrix} \times \begin{vmatrix} (ac_1) & (ad_1) \\ (bd_1) & (cd_1) \end{vmatrix},$$

an expression whose degree is 8 and weight 12, in place of degree 6 and weight 9; whence it appears that it ought to be divisible by a factor whose degree is 2 and weight 3. This factor must therefore be of the form $l\,(bc_1) + m\,(ad_1)$. We proceed now to show that it is (ad_1); and to find the quotient when this factor is removed.

For this purpose, retaining only the terms which do not directly involve (ad_1), we have

$$(ab_1)(cd_1)\{(ab_1)(cd_1) + (ca_1)(bd_1)\},$$

which is divisible by (ad_1), since

$$(bc_1)(ad_1) + (ca_1)(bd_1) + (ab_1)(cd_1) \equiv 0.$$

Expanding the determinants, and dividing off by (ad_1), we find ultimately the quotient

$$\begin{gathered}(ad_1)^3 - 2\,(ab_1)\,(cd_1)\,(ad_1) + (bd_1)\,(ca_1)\,(ad_1)\\ + (ca_1)^2\,(cd_1) + (ab_1)\,(bd_1)^2 - (ab_1)\,(bc_1)\,(dc_1),\end{gathered}$$

which, being of the proper degree and weight, is the resultant.

If we proceed in a similar manner to form the resultant of two biquadratic equations, by reducing the process to an elimination between two cubic equations, we shall have to remove an extraneous factor of the fourth degree, which is the condition that these cubics should have a common factor when the biquadratics from which they are derived have not necessarily a common factor; and in general, if we seek by this method the resultant of two equations of the n^{th} degree, eliminating between two equations of the $(n-1)^{th}$ degree, we shall have to remove

an extraneous factor of the order $2n-4$. This method therefore is inferior to all the preceding methods; and it cannot be conveniently used except when, from the nature of the investigation, extraneous factors can be easily removed.

156. **Discriminants.**—The *discriminant* of an equation involving a single unknown is the simplest function of the coefficients, in a rational and integral form, whose vanishing expresses the condition for equal roots. We have had examples of such functions in Arts. 43 and 68. We proceed to show that they come under eliminants as particular cases. If an equation $f(x)=0$ has a double root, this root must occur once in the equation $f'(x)=0$; and subtracting $xf'(x)$ from $nf(x)$, the same root must occur in the equation $nf(x)-xf'(x)=0$.

This is an equation of the $(n-1)^{th}$ degree in x; and by eliminating x between it and the equation $f'(x)=0$, which is also of the $(n-1)^{th}$ degree, we obtain a function of the coefficients whose vanishing expresses the condition for equal roots. The degree of this eliminant in the coefficients of $f(x)$ is $2(n-1)$; and its weight is $n(n-1)$, as may be seen by examining the specimen terms given in section (1), Art. 151. Expressed as a symmetric function of the roots of the given equation, the discriminant will be the product of all the differences in the lowest power which can be expressed in a rational form in terms of the coefficients. Now the product of the squares of the differences $\Pi(\alpha_1-\alpha_2)^2$ can be so expressed; and since it is of the $2(n-1)^{th}$ degree in any one root, and of the $n(n-1)^{th}$ degree in all the roots, it follows that the discriminant multiplied by a numerical factor is equal to $a_0^{2(n-1)}\,\Pi(\alpha_1-\alpha_2)^2$.

If the function $f(x)$ be made homogeneous by the introduction of a second variable y, the two functions whose resultant is the discriminant of $f(x)$ are the differential coefficients of $f(x)$ with regard to x and y respectively. In the same way, in general, the discriminant of a function homogeneous in any number n of variables is the result of eliminating the variables from the n equations obtained by differentiating with regard to each variable in turn.

Examples.

1. Find the discriminant of

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

We have here to find the eliminant of the two equations

$$a_0x^2 + 2a_1x + a_2 = 0,$$
$$a_1x^2 + 2a_2x + a_3 = 0.$$

The condition for a common root is, by Art. 149,

$$4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) - (a_0a_3 - a_1a_2)^2 = 0.$$

The function of the coefficients here obtained is therefore the discriminant, which may also be written in the form of a determinant, as follows, by Art. 153 :

$$\begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix}.$$

It can be easily verified that this value of the discriminant is the same as that already obtained in Art. 42.

2. Express as a determinant the discriminant of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

We have here to eliminate x from the equations

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$
$$a_1x^3 + 3a_2x^2 + 3a_3x + a_4 = 0.$$

By the method of Art. 153 the resultant is

$$\begin{vmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ a_1 & 3a_2 & 3a_3 & a_4 & 0 & 0 \\ 0 & a_1 & 3a_2 & 3a_3 & a_4 & 0 \\ 0 & 0 & a_1 & 3a_2 & 3a_3 & a_4 \end{vmatrix}.$$

This must be the same as $I^3 - 27J^2$ of Art. 68.

3. Express the discriminant of the quartic as a determinant by Bezout's method of elimination.

4. Prove that the discriminant, Δ_m, of the equation

$$U \equiv ax^m + by^m + cz^m = 0,$$

where
$$x + y + z \equiv 0,$$

may be obtained by rendering rational, in the form $\Delta_m = 0$, the equation

$$(bc)^{\frac{1}{m-1}} + (ca)^{\frac{1}{m-1}} + (ab)^{\frac{1}{m-1}} = 0\,;$$

and calculate in particular the values of Δ_3, Δ_4, Δ_5.

When z is replaced by its value from $x + y + z = 0$ the given function U contains two variables, and the discriminant is obtained by eliminating x and y from

$$\frac{dU}{dx} = 0 \text{ and } \frac{dU}{dy} = 0.$$

5. Prove by elimination that $J = 0$ is one condition for the equality of three roots of the biquadratic of Ex. 2.

Since the triple root must be a double root of

$$U_3 \equiv a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$

and therefore a single root of $a_1x^2 + 2a_2x + a_3 = 0$; and since it must also be a single root of

$$U_2 \equiv a_0x^2 + 2a_1x + a_2 = 0,$$

it follows from the identity

$$U_4 \equiv x^2U_2 + 2x(a_1x^2 + 2a_2x + a_3) + a_2x^2 + 2a_3x + a_4$$

that the triple root must be a root common to the three equations

$$a_0x^2 + 2a_1x + a_2 = 0,$$

$$a_1x^2 + 2a_2x + a_3 = 0,$$

$$a_2x^2 + 2a_3x + a_4 = 0.$$

Hence the condition

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \equiv J = 0.$$

That the other condition for a triple root is $I = 0$ may be inferred from Ex. 10, p. 338; for when three roots are equal the discriminant must vanish, and it is of the form $lI^3 + mJ^2$.

6. Prove that the discriminant of the product of two functions is the product of their discriminants multiplied by the square of their eliminant.

This appears by applying the results of Art. 150 and the present Article; for the product of the squares of the differences of all the roots is made up of the product of the squares of the differences of the roots of each equation separately and the square of the product of the differences formed by taking each root of one equation with all the roots of the other.

157. **Determination of a Root common to two Equations.**—If R be the resultant of two equations

$$U \equiv a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 = 0,$$
$$V \equiv b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 = 0,$$

and a any common root, then

$$a = \frac{\dfrac{dR}{da_1}}{\dfrac{dR}{da_0}} = \frac{\dfrac{dR}{da_2}}{\dfrac{dR}{da_1}} = \frac{\dfrac{dR}{da_3}}{\dfrac{dR}{da_2}} = \&\text{c}.$$

To prove this we first show that functions $\phi(x)$ and $\psi(x)$ can be obtained such that $R \equiv U\phi(x) + V\psi(x)$, namely, when U and V are multiplied by $\phi(x)$ and $\psi(x)$, respectively, and added, all terms involving x vanish identically. Take, for example, the form of R given for two functions of the 4th and 3rd degrees, respectively, in Ex. 2, Art. 153. Multiply the second column by x, the third by x^2, &c., and add to the first column, thus obtaining U, xU, x^2U, V, xV, x^2V, x^3V for the constituents of the first column. The determinant when expanded takes then the form $U\phi(x) + V\psi(x)$, where ϕ is a quadratic function, and ψ a cubic function of x. This mode of proof can be applied to any two functions; and it will be observed in the general case that ϕ and ψ are of the degrees $n - 1$ and $m - 1$, respectively, the degrees of U and V being m and n. We have therefore

$$R \equiv U\phi + V\psi;$$

whence

$$\frac{dR}{da_p} \equiv x^p\phi + U\frac{d\phi}{da_p} + V\frac{d\psi}{da_p},$$

$$\frac{dR}{da_{p+1}} \equiv x^{p+1}\phi + U\frac{d\phi}{da_{p+1}} + V\frac{d\psi}{da_{p+1}};$$

and when a is a common root of the equations $U = 0$, and $V = 0$, we have, substituting this value for x in the preceding equations,

$$a\frac{dR}{da_p} = \frac{dR}{da_{p+1}},$$

which proves the proposition.

A double root of an equation can be determined in a similar manner by differentiating the discriminant Δ.

When the equations $U = 0$ and $V = 0$ have two roots common, the first differential coefficients of R with regard to a_p, a_{p+1}, &c., vanish identically, and it is necessary to proceed to a second differentiation. In this case the common roots are given as the roots of the quadratic equation

$$\frac{d^2R}{da_p^{\,2}}x^2 - 2\frac{d^2R}{da_p\,da_{p+1}}x + \frac{d^2R}{da_{p+1}^{\,2}} = 0,$$

as is easily seen by differentiating the value of R above given, when the first member of the equation last written is found to be equal to

$$\left(\frac{d^2\phi}{da_p^{\,2}}x^2 - 2\frac{d^2\phi}{da_p\,da_{p+1}}x + \frac{d^2\phi}{da_{p+1}^{\,2}}\right)U + \left(\frac{d^2\psi}{da_p^{\,2}}x^2 - 2\frac{d^2\psi}{da_p\,da_{p+1}}x + \frac{d^2\psi}{da_{p+1}^{\,2}}\right)V,$$

and must therefore vanish when either of the common roots is substituted for x.

A similar process will apply if there are three or more common roots.

158. **Symmetric Functions of the Roots of two Equations.**—If it be required to calculate a symmetric function involving the roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m$ of the equation

$$\phi(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0, \qquad (1)$$

along with the roots $\beta_1, \beta_2, \beta_3, \ldots \beta_n$, of the equation

$$\psi(y) \equiv b_0y^n + b_1y^{n-1} + b_2y^{n-2} + \ldots + b_n = 0, \qquad (2)$$

we proceed as follows :—

Assume a new variable t connected with x and y by the equation

$$t = \lambda x + \mu y;$$

and let y be eliminated by means of this equation from (2). The result is an equation of the n^{th} degree in x whose coefficients involve λ, μ, and t in the n^{th} power. Now let x be eliminated by any of the preceding methods from this equation and (1). We obtain an equation of the mn^{th} degree in t, whose roots are the mn values of the expression $\lambda\alpha + \mu\beta$.

If, now, it be required to calculate in terms of the coefficients of $\phi(x)$ and $\psi(y)$ any symmetric function such as $\Sigma\alpha^p\beta^q$, we form the sum of the $(p+q)^{th}$ powers of the roots of the equation in t. We thus find the value of $\Sigma(\lambda\alpha+\mu\beta)^{p+q}$ expressed in terms of the original coefficients and the several powers of λ and μ. The coefficient of $\lambda^p\mu^q$ in this expression will furnish the required value of $\Sigma\alpha^p\beta^q$ in terms of the coefficients of $\phi(x)$ and $\psi(y)$.

If it were required to calculate symmetric functions of the roots of three equations, we should assume

$$t=\lambda x+\mu y+\nu z,$$

eliminate x, y, z, and proceed as before. This method therefore applies whatever the number of equations; and by making the coefficients $a_r=b_r=c_r$, &c., we fall back on the symmetric functions of the roots of a single equation already calculated.

The examples which follow are given to illustrate the principles contained in the foregoing chapter.

Examples.

1. Eliminate x from the equations

$$ax^2+bx+c=0,$$
$$x^3=1.$$

Multiplying the first equation by x, we have, since $x^3=1$,

$$bx^2+cx+a=0;$$

and multiplying again by x, we have

$$cx^2+ax+b=0.$$

Eliminating x^2 and x linearly from these three equations, the result is expressed as a determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0.$$

If the method of symmetric functions (Art. 150) be employed, and the roots of the second equation substituted in the first, the resultant is obtained in the form

$$(a+b+c)(a\omega^2+b\omega+c)(a\omega+b\omega^2+c).$$

2. Eliminate similarly x from the equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$
$$x^5 = 1.$$

The result is a circulant of the fifth order obtained by a process similar to that of the last example. By aid of the method of symmetric functions the five factors can be written down (cf. Ex. 33, p. 304). An analogous process may be applied in general to two equations of this kind.

3. Apply the method of Art. 152 to find the conditions that the two cubics

$$\phi(x) \equiv ax^3 + bx^2 + cx + d = 0,$$
$$\psi(x) \equiv a'x^3 + b'x^2 + c'x + d' = 0$$

should have two common roots.

When this is the case, identical results must be obtained by multiplying $\phi(x)$ by the third factor of $\psi(x)$, and $\psi(x)$ by the third factor of $\phi(x)$. We have, therefore,

$$(\lambda'x + \mu')\,\phi(x) \equiv (\lambda x + \mu)\,\psi(x),$$

where λ, μ, λ', μ' are indeterminate quantities. This identity leads to the equations

$$\begin{aligned}
\lambda'a \qquad\quad - \lambda a' \qquad\quad &= 0,\\
\lambda'b + \mu'a - \lambda b' - \mu a' &= 0,\\
\lambda'c + \mu'b - \lambda c' - \mu b' &= 0,\\
\lambda'd + \mu'c - \lambda d' - \mu c' &= 0,\\
\mu'd \qquad\quad - \mu d' &= 0.
\end{aligned}$$

Eliminating λ', μ', λ, μ from every four of these, we obtain five determinants, whose vanishing expresses the required conditions. There is a convenient notation in use to express the result of eliminating from a number of equations of this kind. In the present instance the vanishing of the five determinants is expressed as follows:—

$$\left\Vert \begin{matrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ a' & b' & c' & d' & 0 \\ 0 & a' & b' & c' & d' \end{matrix} \right\Vert = 0,$$

the determinants being formed by omitting each column in turn.

4. Prove the identity

$$\begin{vmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\alpha' & \alpha\beta' + \alpha'\beta & \beta\beta' \\ \alpha'^2 & 2\alpha'\beta' & \beta'^2 \end{vmatrix} \equiv (\alpha\beta' - \alpha'\beta)^3.$$

This appears by eliminating x and y from the equations

$$\alpha x + \beta y = 0, \qquad \alpha' x + \beta' y = 0\,;$$

for from these equations we derive

$$(\alpha x + \beta y)^2 = 0, \quad (\alpha x + \beta y)(\alpha' x + \beta' y) = 0, \quad (\alpha' x + \beta' y)^2 = 0.$$

The determinant above written is the result of eliminating x^2, xy, and y^2 from the latter equations; and this result must be a power of the determinant derived by eliminating x, y from the linear equations.

5. Prove similarly

$$\begin{vmatrix} \alpha^3 & 3\alpha^2\beta & 3\alpha\beta^2 & \beta^3 \\ \alpha^2\alpha' & \alpha^2\beta' + 2\alpha\alpha'\beta & 2\alpha\beta\beta' + \alpha'\beta^2 & \beta^2\beta' \\ \alpha\alpha'^2 & \alpha'^2\beta + 2\alpha\alpha'\beta' & 2\alpha'\beta\beta' + \alpha\beta'^2 & \beta\beta'^2 \\ \alpha'^3 & 3\alpha'^2\beta' & 3\alpha'\beta'^2 & \beta'^3 \end{vmatrix} \equiv (\alpha\beta' - \alpha'\beta)^6.$$

6. Prove the result of Ex. 13, p. 297, by eliminating λ, μ, λ', μ', from four equations

$$\alpha' = \frac{\lambda\alpha + \mu}{\lambda'\alpha + \mu'}, \quad \beta' = \frac{\lambda\beta + \mu}{\lambda'\beta + \mu'}, \text{ \&c.,}$$

connecting the variables in homographic transformation.

7. Given

$$U \equiv Au^2 + 2Buv + Cv^2,$$

$$V \equiv A'u^2 + 2B'uv + C'v^2,$$

$$u \equiv ax^2 + 2bxy + cy^2,$$

$$v \equiv a'x^2 + 2b'xy + c'y^2,$$

determine the resultant of U and V considered as functions of x, y.

Since

$$U = A(u - \alpha v)(u - \beta v),$$

$$V = A'(u - \alpha' v)(u - \beta' v),$$

if U and V vanish for common values of x, y, some pair of factors, as $u - \alpha v$ and $u - \alpha' v$, must vanish; whence forming the resultant of $u - \alpha v$ and $u - \alpha' v$, and representing the resultant of u and v by $R(u, v)$, we have

$$R(u - \alpha v, \quad u - \alpha' v) = (\alpha - \alpha')^2 R(u, v)\,;$$

and multiplying all these resultants together, we find

$$R(U_x, V_x) = A^4 A'^4 (\alpha - \alpha')^2 (\beta - \beta')^2 (\alpha - \beta')^2 (\beta - \alpha')^2 \{R(u, v)\}^4,$$

or

$$R(U_x, V_x) = \{R(U, V)\}^2 \{R(u, v)\}^4.$$

8. Prove that the equation whose roots are the differences of the roots of a given equation $f(x) = 0$ may be obtained by eliminating x from the equations

$$f(x) = 0, \quad f'(x) + f''(x)\frac{y}{1\,.\,2} + f'''(x)\frac{y^2}{1\,.\,2\,.\,3} + \&c. = 0;$$

and determine the degree of the equation in y (cf. Art. 44).

9. Eliminate x, y, z from the equations

$$x + y + z = 0,$$
$$ayz + bzx + cxy = 0,$$
$$ay^3z^3 + bz^3x^3 + cx^3y^3 = 0.$$

Taking the first two equations along with an assumed linear equation with arbitrary coefficients, viz.,

$$\lambda x + \mu y + \nu z = 0;$$

and eliminating x, y, z, we easily obtain

$$a\lambda^2 + b\mu^2 + c\nu^2 + (a - b - c)\,\mu\nu + (b - c - a)\,\nu\lambda + (c - a - b)\,\lambda\mu = 0, \qquad (1)$$

which must be equivalent to the equation

$$(\lambda x_1 + \mu y_1 + \nu z_1)(\lambda x_2 + \mu y_2 + \nu z_2) = 0, \qquad (2)$$

where x_1, y_1, z_1, x_2, y_2, z_2 are the two systems of values of x, y, z common to the first two of the given equations. Substituting these values in the third of the given equations, we have

$$R = (ay_1^3z_1^3 + bz_1^3x_1^3 + cx_1^3y_1^3)(ay_2^3z_2^3 + bz_2^3x_2^3 + cx_2^3y_2^3);$$

and reducing this value of R by means of the symmetric functions determined by the comparison of the equations (1) and (2), we find

$$4R = 4p^2q + q^2 + 27pr,$$

where
$$p = a^2 + b^2 + c^2 - 2bc - 2ca - 2ab,$$
$$q = abc\,(a + b + c),$$
$$r = a^2b^2c^2.$$

10. If U, V, W are three given functions of x of the degrees m, n, $m + n - 1$, respectively, prove that an identical relation exists of the form

$$RW \equiv U\phi(x) + V\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are functions to be determined, of the degrees $n - 1$ and $m - 1$, respectively, and R is the resultant of U and V.

11. Verify the results of Art. 157 by differentiating the value of R given in Art. 150.

CHAPTER XV.

COVARIANTS AND INVARIANTS.

159. **Definitions.**—In this and the following chapters the notation

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

will be employed to represent the quantic

$$a_0 x^n + n a_1 x^{n-1} y + \frac{n(n-1)}{1 \cdot 2} a_2 x^{n-2} y^2 + \ldots + n a_{n-1} x y^{n-1} + a_n y^n,$$

a homogeneous function of x and y, written with binomial coefficients. If we put $y = 1$, this quantic becomes U_n of Art. 35.

Let ϕ be a seminvariant, that is, a rational, integral, and homogeneous symmetric function, of the order ϖ, of the roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ of the equation $U_n \equiv (a_0, a_1, a_2 \ldots a_n)(x, 1)^n = 0$, this function involving only the differences of the roots (see Art. 147); then if

$$\frac{1}{\alpha_1 - x}, \quad \frac{1}{\alpha_2 - x}, \ldots \frac{1}{\alpha_n - x}$$

be substituted for $\alpha_1, \alpha_2, \ldots \alpha_n$, respectively, the result multiplied by U_n^{ϖ} (to remove fractions) is a *covariant* of U_n if it involves x, and an *invariant* if it does not involve x.

From this definition of an invariant we may infer at once that

$$a_0^{\varpi} \phi(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n)$$

is an invariant of U_n when ϕ is composed of a number of terms of the same type, each of which involves all the roots, and each root in the same degree ϖ.

These definitions may be extended to the case where ϕ (the function of differences) involves symmetrically the roots of several equations $U_p = 0$, $U_q = 0$, $U_r = 0$, &c., the roots of these equations entering ϕ in the orders ϖ, ϖ', ϖ'', &c. . . . respectively. We may substitute for each root a, $\dfrac{1}{a-x}$ as before, and remove fractions by the multiplier $U_p^{\varpi} U_q^{\varpi'} U_r^{\varpi''} \ldots$ &c. If the result involves the variable x, we obtain a covariant of the system of quantics U_p, U_q, U_r, &c.; and if it does not, ϕ is an invariant of the system.

160. **Formation of Covariants and Invariants.**—We proceed now to show how the foregoing transformations may be conveniently effected, and covariants and invariants calculated in terms of the coefficients. With this object, let the seminvariant be expressed in terms of the coefficients as follows:—

$$a_0^{\varpi}\phi(a_1, a_2, \ldots a_n) = F(a_0, a_1, a_2, \ldots a_n).$$

Now, changing the roots into their reciprocals, and consequently a_0 into a_n, &c., a_r into a_{n-r}, &c. (that is, giving the suffixes their complementary values), we have

$$a_0^{\varpi}\psi(a_1, a_2, \ldots a_n) = F(a_n, a_{n-1}, \ldots a_0),$$

where ψ is an integral symmetric function of the roots, and F the corresponding value in terms of the coefficients. This function is called the *source** of the covariant derived therefrom.

Again, substituting $a_1 - x$, $a_2 - x$, . . . $a_n - x$ for a_1, a_2, . . . a_n, and consequently U_r, &c., for a_r, &c. (see Art. 35), we find

$$a_0^{\varpi}\psi(a_1 - x, a_2 - x, \ldots a_n - x) = F(U_n, U_{n-1}, \ldots U_1, U_0).$$

Thus, by two steps we derive a covariant from a function of the differences, and find at the same time its equivalent calculated in terms of the coefficients.

To illustrate this mode of procedure we take the example in the case of the cubic

$$a_0^2\Sigma(a-\beta)^2 = 18(a_1^2 - a_0a_2);$$

* This term was introduced by Mr. Roberts.

whence, changing the roots into their reciprocals, and a_0, a_1, a_2, a_3 into a_3, a_2, a_1, a_0, we have

$$a_0^2 \Sigma \alpha^2 (\beta - \gamma)^2 = 18 (a_2^2 - a_3 a_1).$$

Again, changing α, β, γ into $\alpha - x$, $\beta - x$, $\gamma - x$, and a_1, a_2, a_3 into U_1, U_2, U_3, respectively, we find

$$a_0^2 \Sigma (\beta - \gamma)^2 (x - \alpha)^2 = 18 (U_2^2 - U_3 U_1).$$

The second member of this equation becomes when expanded

$$U_1 U_3 - U_2^2 \equiv (a_0 a_2 - a_1^2) x^2 + (a_0 a_3 - a_1 a_2) x + (a_1 a_3 - a_2^2).$$

This covariant is called the *Hessian* of U_3. We refer to it as H_x, since H is its leading coefficient.

As a second example we take the following function of the quartic :—

$$a_0^2 \Sigma (\beta - \gamma)^2 (\alpha - \delta)^2 = 24 (a_0 a_4 - 4 a_1 a_3 + 3 a_2^2) ; \qquad (1)$$

whence, changing the roots into their reciprocals, and a_0, a_1, a_2, a_3, a_4 into a_4, a_3, a_2, a_1, a_0, we have

$$a_0^2 \Sigma (\gamma - \beta)^2 (\delta - \alpha)^2 = 24 (a_4 a_0 - 4 a_3 a_1 + 3 a_2^2).$$

These transformations, therefore, do not alter equation (1) : again, since in this case $\psi(\alpha, \beta, \gamma, \delta)$ is a function of the differences of the roots, ψ is unchanged when $\alpha - x$, $\beta - x$, &c. . . . , are substituted for $\alpha, \beta, \gamma, \delta$. We infer that $a_0 a_4 - 4 a_1 a_3 + 3 a_2^2$ is an invariant of the quartic U_4.

We observe also, in accordance with what was stated in Art. 159, since

$$\phi \equiv (\beta - \gamma)^2 (\alpha - \delta)^2 + (\gamma - \alpha)^2 (\beta - \delta)^2 + (\alpha - \beta)^2 (\gamma - \delta)^2,$$

that any one of the three terms of which ϕ is made up involves each of the roots in the degree ϖ, which is here equal to 2.

In a similar manner it may be shown that

$$\begin{aligned} a_0^3 \{(\gamma - \alpha)(\beta - \delta) - (\alpha - \beta)(\gamma - \delta)\} \{(\alpha - \beta)(\gamma - \delta) - (\beta - \gamma)(\alpha - \delta)\} \\ \times \{(\beta - \gamma)(\alpha - \delta) - (\gamma - \alpha)(\beta - \delta)\} \\ = -432 (a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3) \end{aligned}$$

is an invariant of the quartic.

There is no difficulty in determining in any particular case whether ϕ leads to an invariant or covariant, for if ϕ leads to an invariant, $\phi = \pm\psi$, that is ϕ is unchanged (except in sign, when its type-term is the product of an odd number of differences of the roots, *i.e.* when its weight is odd) when for the roots their reciprocals are substituted, and fractions removed by the simplest multiplier $(a_1a_2a_3 \ldots a_n)^{\varpi}$. An invariant whose weight is odd is called a *skew invariant*.

161. **Properties of Covariants and Invariants.**— Since ϕ is a homogeneous function of the roots, the covariant derived from it may be written under the form

$$\frac{U^{\varpi}}{x^{\kappa}}\phi\left(\frac{x}{a_1-x}, \quad \frac{x}{a_2-x}, \ldots \frac{x}{a_n-x}\right),$$

where ϖ is the order, and κ the weight of ϕ.

Also, as ϕ is a function of the differences, we may add 1 to each constituent such as $\dfrac{x}{a_r-x}$, thus obtaining $\dfrac{a_r}{a_r-x}$. Again, multiplying each constituent by x, the covariant becomes

$$\frac{U^{\varpi}}{x^{2\kappa}}\phi\left(\frac{a_1x}{a_1-x}, \quad \frac{a_2x}{a_2-x}, \ldots \frac{a_nx}{a_n-x}\right).$$

Employing now the notation x', a_1', a_2', &c., for the reciprocals of x, a_1, a_2, &c.; and denoting by U' the function whose roots are a_1', a_2', ... a_n', viz.

$$U' \equiv a_nx'^n + na_{n-1}x'^{n-1} + \&c., \ldots + na_1x' + a_0 = 0;$$

since $$\frac{1}{a_r' - x'} = \frac{-a_rx}{a_r - x},$$

and $$U = a_nx^n(x'-a_1')(x'-a_2')\ldots(x'-a_n') = x^nU',$$

the covariant above written is easily reduced to the form

$$(-1)^{\kappa}x^{n\varpi-2\kappa}U'^{\varpi}\phi\left(\frac{1}{a_1'-x'}, \quad \frac{1}{a_2'-x'}, \ldots \frac{1}{a_n'-x'}\right);$$

whence it is proved that the covariant is unaltered when for x, a_1, a_2, ... a_n their reciprocals are substituted, and the result

multiplied by $(-1)^\kappa x^{n\varpi-2\kappa}$. This transformation changes a_r into a_{n-r}, that is, each coefficient into the coefficient with complementary suffix.

Now if any covariant whose degree is m be written in the form

$$(B_0, B_1, B_2, \ldots B_m)(x, 1)^m; \qquad (1)$$

changing $a_0, a_1, \ldots a_n, x$, into $a_n, a_{n-1}, \ldots a_0, \frac{1}{x}$, we have another form for this covariant, namely,

$$(-1)^\kappa x^{n\varpi-2\kappa}(C_0, C_1, C_2, \ldots C_m)\left(\frac{1}{x}, 1\right)^m;$$

and as this form is an integral function of x of the same type as (1), we have, by comparing the two forms,

$$m = n\varpi - 2\kappa, \quad B_0 = (-1)^\kappa C_m, \ldots B_r = (-1)^\kappa C_{m-r};$$

thus determining the degree of the covariant in terms of the order and weight of the function ϕ, and showing that the conjugate coefficients (*i. e.* those equally removed from the extremes) are related in the following way :—

If $F(a_0, a_1, a_2, \ldots a_n)$ *be any coefficient of the covariant*, $(-1)^\kappa F(a_n, a_{n-1}, a_{n-2}, \ldots a_0)$ *is its conjugate.*

This property is characteristic of covariants, and is not possessed by semicovariants, although the two classes of functions agree in the mode of formation by the operator D as will appear in the Article which follows (cf. Art. 147).

From the expression for the degree of a covariant in terms of ϖ and κ, namely $n\varpi - 2\kappa$, we may draw the following important inferences :—

(1). *If* $a_0^{\varpi}\phi$ *is an invariant*, $n\varpi = 2\kappa$.

For, in this case ϕ and ψ are the same function, and consequently their weights κ and $n\varpi - \kappa$ also the same.

(2). *All the invariants of quantics of odd degrees are of even order.*

For if n be odd, it is plain from the equation $n\varpi = 2\kappa$ that ϖ must be even, and κ a multiple of n.

(3). *All covariants of quantics of even degrees are of even degrees.*

For in this case $n\varpi - 2\kappa$ is even.

(4). *Covariants of quantics of odd degrees are of odd or even degree according as the order of their coefficients is odd or even.*

(5). *The resultant of two covariants is always of an even degree in the coefficients of the original quantic.*

For, the degree of the resultant expressed in terms of the orders and weights of the covariants is

$$\varpi(n\varpi' - 2\kappa') + \varpi'(n\varpi - 2\kappa) = 2(n\varpi\varpi' - \varpi\kappa' - \varpi'\kappa).$$

162. **Formation of Covariants by the Operator D.**—From Art. 147 we infer that the expansion of $F(U_n, U_{n-1}, \ldots U_0)$ may be expressed by means of the Differential Calculus in the form

$$F_0 + xDF_0 + \frac{x^2}{1.2}D^2F_0 + \ldots + \frac{x^r}{1.2.3\ldots r}D^rF_0 + \ldots,$$

where F_0 is the result of making $x = 0$ in $F(U_n, U_{n-1}, \ldots U_0)$, viz.

$$F_0 = F(a_n, a_{n-1}, \ldots a_0),$$

and $$D = a_0\frac{d}{da_1} + 2a_1\frac{d}{da_2} + 3a_2\frac{d}{da_3} + \ldots + na_{n-1}\frac{d}{da_n}.$$

In forming a covariant by this process, the source F_0 with which we set out is altered by the successive operations D, each operation reducing the weight by one, till we arrive at the original function $F(a_0, a_1, \ldots a_n)$ from which the source was formed. Since this is a function of the differences, the expression resulting from the next operation D vanishes, and the covariant is completely formed. The corresponding operations δ on the symmetric function ψ have the effect of reducing the degree in the roots by one each step, the final symmetric function containing the differences only. Thus by successive operations we obtain two expressions for a covariant—one in terms of the roots, and the other in terms of the coefficients.

The degree m of the covariant is plainly equal to the number of times δ operates in reducing ψ_0 to ϕ, *i.e.* equal to the difference of the weights of the extreme coefficients. And since

$$\psi_0 = (a_1 a_2 \ldots a_n)^{\varpi} \phi\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_n}\right),$$

the weight of ψ_0 is $n\varpi - \kappa$, where κ is the weight of $\phi(a_1, a_2, \ldots a_n)$; hence the degree of the covariant whose leading coefficient is $a_0^{\varpi}\phi$ is $n\varpi - 2\kappa$, the same value as before obtained. We add some simple examples in illustration of this method.

Examples.

1. Form the Hessian of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

Taking the function $H \equiv a_0a_2 - a_1^2$, we find, as in Art. 160,

$$a_0^2\Sigma\alpha^2(\beta - \gamma)^2 = 18(a_2^2 - a_1a_3).$$

Operating on the left-hand side by δ, and on the right-hand side by D, we obtain

$$-a_0^2\Sigma 2\alpha(\beta - \gamma)^2 = 18(a_1a_2 - a_0a_3);$$

and operating in the same way again,

$$a_0^2\Sigma 2(\beta - \gamma)^2 = 36(a_1^2 - a_0a_2).$$

The next operation causes both sides of the equation to vanish. Hence the required covariant is, as in Art. 160,

$$(a_1a_3 - a_2^2) + (a_0a_3 - a_1a_2)x + (a_0a_2 - a_1^2)x^2.$$

We find at the same time the corresponding expression in terms of x and the roots.

2. Form the Hessian of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

The covariant whose leading coefficient is $H \equiv a_0a_2 - a_1^2$ is called the Hessian of the biquadratic. Its degree is 4, since $\varpi = 2$, and $\kappa = 2$; and $\therefore n\varpi - 2\kappa = 4$. Changing the coefficients into their complementaries, the source of the covariant is $a_4a_2 - a_3^2$, and we easily find

$$H_x \equiv (a_0a_2 - a_1^2)x^4 + 2(a_0a_3 - a_1a_2)x^3 + (a_0a_4 + 2a_1a_3 - 3a_2^2)x^2$$
$$+ 2(a_1a_4 - a_2a_3)x + (a_2a_4 - a_3^2).$$

3. Form for a cubic a covariant whose leading coefficient is the semi-invariant G.

Changing the coefficients in G into their complementaries, we get the source $a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3$, and operating with D we easily obtain the covariant in the following form:—

$$(a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3) + 3(a_3a_2a_0 + a_2^2a_1 - 2a_3a_1^2)x$$
$$-3(a_0a_1a_3 + a_1^2a_2 - 2a_0a_2^2)x^2 - (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)x^3.$$

In this the conjugate coefficients (see Art. 161) differ in sign as well as in the interchange of complementaries, the weight of G being odd. The student will have no difficulty in expressing this covariant in terms of x and the roots by the aid of the value of G given in Ex. 15, Art. 27.

163. **Theorem.**—*Any function of the differences of the roots of a covariant or semicovariant is a function of the differences of the roots of the original equation.*

Let the covariant or semicovariant be

$$\phi(x) \equiv (x - \rho_1)(x - \rho_2) \ldots (x - \rho_p).$$

Since ϕ is a function of the differences of $x, a_1, a_2, \ldots a_n$, we have

$$\frac{d\phi}{dx} - \delta\phi \equiv 0;$$

and substituting in this identical equation each root ρ_1, ρ_2, &c. in succession, we easily prove $\delta\rho_1 + 1 = 0$, $\delta\rho_2 + 1 = 0, \ldots \delta\rho_j + 1 = 0$, &c.; whence

$$\delta(\rho_j - \rho_k) = 0,$$

which proves the theorem.

In the preceding pages many instances have been given in which the roots of covariants or semicovariants are expressed in terms of the roots of the original equation; and the student will easily verify that the result of the operation of δ on any such expression is -1. The roots of the covariants in Exs. 1 and 3 of the preceding Article are given in Ex. 25, p. 57, and Ex. 13, p. 88, respectively; and roots of semicovariants will be found in Exs. 10, 11, p. 87, and Exs. 12, 14, p. 88.

The theorem here proved is clearly true also for any function of the differences of the roots of two or more covariants or semicovariants.

164. **Homographic Transformation applied to the Theory of Covariants.**—Hitherto we have discussed the theory of covariants and invariants through the medium of the roots of equations. We proceed now to give some account of a different and more general mode of treatment, by means of which this theory may be extended to quantics homogeneous in more than two variables, such as present themselves in the numerous important geometrical applications of the theory. Although this enlarged view of the subject does not come within the scope of the present work, we think it desirable to show the connexion between the method of treatment we have adopted and the more general method referred to. With this object we give in the present Article two important propositions.

Prop. I.—*Let any quantic U_n be transformed by the homographic transformation*

$$x = \frac{\lambda x' + \mu}{\lambda' x' + \mu'};$$

if I and I' be corresponding invariants of the two forms, we have

$$I' = (\lambda\mu' - \lambda'\mu)^{\kappa} I.$$

To prove this, let

$$I = a_0^{\varpi} \Sigma (a_1 - a_2)^a (a_2 - a_3)^b \ldots (a_1 - a_n)^l,$$

each root entering in the degree ϖ.

Now, transforming the similar value of I', since $x' = \dfrac{\mu' x - \mu}{\lambda - \lambda' x}$, we have

$$a_p' - a_q' = \frac{(\lambda\mu' - \lambda'\mu)(a_p - a_q)}{(\lambda - \lambda' a_p)(\lambda - \lambda' a_q)}.$$

Again, transforming U_n, and rendering the result integral, U_n' takes the form

$$a_0'(x' - a_1')(x' - a_2') \ldots (x' - a_n'),$$

where

$$a_0' = a_0(\lambda - \lambda' a_1)(\lambda - \lambda' a_2) \ldots (\lambda - \lambda' a_n);$$

making these substitutions for all the differences, and for a_0', the denominators of the fractions which enter by the transformation disappear; and we have, finally,

$$I' = (\lambda\mu' - \lambda'\mu)^{\kappa} I.$$

PROP. II.—*If* $\phi(x)$ *be a covariant of the quantic* U_n, *the new value of* $\phi(x)$, *after homographic transformation, is (when rendered integral)*

$$(\lambda\mu' - \lambda'\mu)^{\kappa} \phi(x).$$

The proof is similar to that of the preceding proposition. We have

$$\phi(x) = a_0^{\varpi} \Sigma (a_1 - a_2)^a (a_2 - a_3)^b \ldots . (x - a_1)^p (x - a_2)^q \ldots,$$

this expression being obtained by substituting

$$x - a_1, \quad x - a_2, \ldots x - a_n \text{ for } a_1, a_2, \ldots a_n$$

in the source of the covariant $\phi(x)$ expressed in terms of the roots. Now, transforming, as in the previous proposition, the value of $\phi(x)$ thus derived; since the factors $\lambda - \lambda' a_1$, $\lambda - \lambda' a_2, \ldots$ all enter in the same degree ϖ in the denominator (for each root enters the source in the degree ϖ), they will all be removed by the multiplier $a_0'^{\varpi}$, and the transformed value of $\phi(x)$ is

$$(\lambda\mu' - \lambda'\mu)^{\kappa} \phi(x).$$

165. **Reduction of Homographic Transformation to a Double Linear Transformation.**—With a view to this reduction let the quantic be written under the homogeneous form

$$U_n = a_0 x^n + n a_1 x^{n-1} y + \frac{n(n-1)}{1 \cdot 2} a_2 x^{n-2} y^2 + \ldots + a_n y^n;$$

and, in place of putting as before $x = \dfrac{\lambda x' + \mu}{\lambda' x' + \mu'}$, and removing fractions to make U_n integral, let now $\dfrac{x}{y} = \dfrac{\lambda x' + \mu y'}{\lambda' x' + \mu' y'}$, where $\dfrac{x}{y}$

and $\frac{x'}{y'}$ are the variables in the ordinary sense. The transformation may therefore be reduced to a linear transformation of both the variables x and y, and can be effected by substituting in the original quantic for x and y, respectively,

$$\lambda x' + \mu y', \quad \lambda' x' + \mu' y',$$

the introduction of fractions being in this way avoided.

Thus we pass from a homographic transformation of functions of a single variable to the linear transformation of homogeneous functions of two variables.

The determinant $\lambda\mu' - \lambda'\mu$, whose constituents are the coefficients which enter into the transformation, is called the *modulus of transformation.*

We are now enabled to restate Propositions I. and II. of Art. 164, in the following way :—

Prop. I.—*An invariant is a function of the coefficients of a quantic, such that when the quantic is transformed by linear transformation of the variables, the same function of the new coefficients is equal to the original function multiplied by a power of the modulus of transformation.*

Prop. II.—*A covariant is a function of the coefficients of a quantic, and also of the variables, such that when the quantic is transformed by linear transformation, the same function of the new variables and coefficients is equal to the original function multiplied by a power of the modulus of transformation.*

The definitions contained in the preceding propositions are plainly applicable to quantics homogeneous in any number of variables, and form the basis of the more extended theory of covariants and invariants referred to in the preceding Article. We give among the following examples an application in the case of a quantic involving three variables.

Examples.

1. Performing the linear transformation

$$x = \lambda X + \mu Y, \quad y = \lambda_1 X + \mu_1 Y,$$

if

$$ax^2 + 2bxy + cy^2 = AX^2 + 2BXY + CY^2,$$

prove that

$$AC - B^2 = (\lambda\mu_1 - \lambda_1\mu)^2 (ac - b^2).$$

2. Performing the same transformation, if

$$(a, b, c, d, e)(x, y)^4 = (A, B, C, D, E)(X, Y)^4,$$

prove that

$$AE - 4BD + 3C^2 = (\lambda\mu_1 - \lambda_1\mu)^4 (ae - 4bd + 3c^2).$$

3. Performing the same transformation, if

$$ax^2 + 2bxy + cy^2 = AX^2 + 2BXY + CY^2,$$

and

$$a_1x^2 + 2b_1xy + c_1y^2 = A_1X^2 + 2B_1XY + C_1Y^2,$$

prove that

$$AC_1 + A_1C - 2BB_1 = (\lambda\mu_1 - \lambda_1\mu)^2 (ac_1 + a_1c - 2bb_1).$$

This follows from Ex. 1, applied to the quadratic forms

$$(a + \kappa a_1)x^2 + 2(b + \kappa b_1)xy + (c + \kappa c_1)y^2 = (A + \kappa A_1)X^2 + 2(B + \kappa B_1)XY + (C + \kappa C_1)Y^2$$

by comparing the coefficients of κ on both sides.

Whence we may infer that, if two quadratics determine a harmonic system, the new quadratics obtained by linear transformation also form an harmonic system. For their roots being α, β and α_1, β_1, we have

$$aa_1\{(\alpha - \alpha_1)(\beta - \beta_1) + (\alpha - \beta_1)(\beta - \alpha_1)\} = 2(ac_1 + a_1c - 2bb_1).$$

4. If the homogeneous quadratic function of three variables

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

be transformed into

$$AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY$$

by the linear substitution

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z, \quad y = \lambda_2 X + \mu_2 Y + \nu_2 Z, \quad z = \lambda_3 X + \mu_3 Y + \nu_3 Z;$$

prove the relation

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = (\lambda_1\mu_2\nu_3)^2 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

where the determinant $(\lambda_1\mu_2\nu_3)$ is the modulus of transformation.

This is easily verified by multiplying the proposed determinant of the original coefficients twice in succession by the modulus of transformation written in the form

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix},$$

and comparing the constituents of the resulting determinant with the expanded values of the coefficients of X^2, Y^2, &c., in the new form.

It appears therefore that the determinant here treated is an invariant of the given function of three variables.

166. **Properties of Covariants derived from Linear Transformation.**—We proceed now to show, taking the second proposition of Art. 165 as the definition of a covariant, that the law of derivation of the coefficients given in Art. 162 immediately follows: that is, *given any one coefficient, all the rest may be determined.*

For this purpose, performing the linear transformation

$$x = X + hY, \quad y = 0X + Y,$$

whose modulus is unity, the quantic

$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$ becomes $(A_0, A_1, A_2, \ldots A_n)(X, Y)^n$, where

$$A_0 = a_0, \; A_1 = a_1 + a_0 h. \; A_2 = a_2 + 2a_1 h + a_0 h^2, \text{ \&c.} \quad \text{(See Art. 35.)}$$

Now, if $\phi(a_0, a_1, a_2, \ldots a_n, x, y)$ be any covariant of this quantic, we have by the definition

$$\phi(a_0, a_1, a_2, \ldots a_n, x, y) = \phi(A_0, A_1, A_2, \ldots A_n, X, Y),$$

or

$$\phi(a_0, a_1, a_2, \ldots a_n, x, y) \equiv \phi(A_0, A_1, A_2, \ldots A_n, x - hy, y).$$

Expanding the second member of this equation, and confining our attention to the terms which multiply h; observing also that $\frac{dAr}{dh} = ra_{r-1}$ when terms are omitted which would be multiplied in the result by h^2, h^3, &c., we have

$$\phi + h\left(-y\frac{d\phi}{dx} + D\phi\right) + h^2\left(\qquad \qquad \right) + \text{\&c.} \ldots \equiv \phi,$$

which must hold whatever value h may have; hence

$$y\frac{d\phi}{dx} = a_0\frac{d\phi}{da_1} + 2a_1\frac{d\phi}{da_2} + 3a_2\frac{d\phi}{da_3} + \ldots + na_{n-1}\frac{d\phi}{da_n}, \qquad (1)$$

and, substituting for ϕ the value

$$(B_0, B_1, B_2, \ldots B_m)(x, y)^m,$$

we have

$$mB_0x^{m-1}y + m(m-1)B_1x^{m-2}y^2 + \ldots + mB_{m-1}y^m$$

$$= DB_0x^m \quad + mDB_1x^{m-1}y \qquad + \ldots + DB_my^m;$$

whence, comparing coefficients, we have the following equations:

$$DB_0 = 0, \quad DB_1 = B_0, \quad DB_2 = 2B_1, \ldots DB_m = mB_{m-1},$$

which determine the law of derivation of the coefficients from the source B_m; the leading coefficient B_0 being a function of the differences, since $DB_0 = 0$.

The calculation of the coefficients is facilitated by the following theorem which has been proved already on different principles:—

Two coefficients of a covariant equally removed from the extremes become equal (plus or minus) when in either of them $a_0, a_1, \ldots a_n$ *are replaced by* $a_n, a_{n-1}, \ldots a_0$, *respectively.*

To prove this, let the quantic be transformed by the linear substitution

$$x = 0X + Y, \quad y = X + 0Y, \quad \text{whose modulus} = -1.$$

Thus

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n = (a_n, a_{n-1}, a_{n-2}, \ldots a_0)(X, Y)^n,$$

and, by definition, any covariant

$$\phi(a_n, a_{n-1}, a_{n-2}, \ldots a_0, X, Y) = (-1)^\kappa \phi(a_0, a_1, a_2 \ldots a_n, x, y)$$

$$= (-1)^\kappa \phi(a_0, a_1, a_2, \ldots a_n, Y, X);$$

whence it follows that the coefficients of the covariant equally removed from the extremes are similar in form, and become identical (except in sign when κ is odd) when for the suffixes their complementary values are substituted.

It is easily inferred in a similar manner that a covariant satisfies the differential equation

$$x\frac{d\phi}{dy} = a_n\frac{d\phi}{da_{n-1}} + 2a_{n-1}\frac{d\phi}{da_{n-2}} + 3a_{n-2}\frac{d\phi}{da_{n-3}} + \ldots + na_1\frac{d\phi}{da_0}, \quad (2)$$

as well as the equation (1) already given.

Again, if $\phi(a_0, a_1, a_2, \ldots a_n)$ be an invariant of the quantic, the former transformation of the present Article gives, employing the definition of Art. 165,

$$\phi(a_0, a_1, a_2, \ldots a_n) = \phi(A_0, A_1, A_2, \ldots A_n);$$

and proceeding as before, in the case of a covariant, we prove that an invariant must satisfy both the differential equations

$$a_0\frac{d\phi}{da_1} + 2a_1\frac{d\phi}{da_2} + 3a_2\frac{d\phi}{da_3} + \ldots + na_{n-1}\frac{d\phi}{da_n} = 0,$$

$$a_n\frac{d\phi}{da_{n-1}} + 2a_{n-1}\frac{d\phi}{da_{n-2}} + 3a_{n-2}\frac{d\phi}{da_{n-3}} + \ldots + na_1\frac{d\phi}{da_0} = 0,$$

either of which may be regarded as contained in the other, since if we make the linear transformation $x = Y$, $y = X$ (whose modulus $= -1$), we have from the definition of an invariant

$$\phi(a_n, a_{n-1}, a_{n-2}, \ldots a_0) = (-1)^\kappa\,\phi(a_0, a_1, a_2, \ldots a_n);$$

proving that an invariant is a function of the coefficients of a quantic which does not alter (except in sign if the weight be odd) when the coefficients are written in direct or reverse order.

The relation between invariants and seminvariants, covariants and semicovariants, is now clear. Invariants of the quantic $(a_0, a_1, \ldots a_n)(x, y)^n$ satisfy both the differential equations last written, whereas seminvariants of $(a_0, a_1, \ldots a_n)(x, 1)^n$ satisfy only the first of these equations. In like manner

semicovariants of $(a_0, a_1, \ldots a_n)(x, 1)^n$ satisfy only the first of the differential equations (1) and (2) above written, whereas both are satisfied by covariants.

Having now explained the nature of Covariants and Invariants of quantics, and the connexion between the two modes in which these functions may be discussed, we proceed to prove certain propositions which are of wide application in the formation of the Covariants and Invariants of quantics transformed by a linear substitution. The student who is reading this subject for the first time may pass at once to the next chapter, where the principles already explained are applied to the cases of the quadratic, cubic, and quartic.

167. Prop. I.—*Let any homogeneous quantic of the n^{th} degree $f(x, y)$ become $F(X, Y)$ by the linear transformation*

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y;$$

also let any function u of x, y become U by the same transformation; then we have

$$M^n f\left(\frac{du}{dy}, -\frac{du}{dx}\right) = F\left(\frac{dU}{dY}, -\frac{dU}{dX}\right), \tag{1}$$

where M is the modulus of transformation.

To prove this proposition, solving the equations

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

we have

$$MX = \mu' x - \mu y, \quad MY = -\lambda' x + \lambda y;$$

whence

$$M\frac{dX}{dx} = \mu', \; M\frac{dX}{dy} = -\mu, \; M\frac{dY}{dx} = -\lambda', \; M\frac{dY}{dy} = \lambda.$$

Again,

$$\frac{du}{dx} = \frac{dU}{dX}\frac{dX}{dx} + \frac{dU}{dY}\frac{dY}{dx} = \frac{1}{M}\left(\mu'\frac{dU}{dX} - \lambda'\frac{dU}{dY}\right),$$

$$\frac{du}{dy} = \frac{dU}{dX}\frac{dX}{dy} + \frac{dU}{dY}\frac{dY}{dy} = \frac{1}{M}\left(-\mu\frac{dU}{dX} + \lambda\frac{dU}{dY}\right),$$

which equations may be put under the form

$$\frac{du}{dy} = \lambda\left(\frac{1}{M}\frac{dU}{dY}\right) + \mu\left(-\frac{1}{M}\frac{dU}{dX}\right),$$

$$-\frac{du}{dx} = \lambda'\left(\frac{1}{M}\frac{dU}{dY}\right) + \mu'\left(-\frac{1}{M}\frac{dU}{dX}\right);$$

and since

$$f(\lambda X + \mu Y, \quad \lambda' X + \mu' Y) \equiv F(X, Y),$$

changing X and Y into $\frac{1}{M}\frac{dU}{dY}$, and $-\frac{1}{M}\frac{dU}{dX}$, respectively, the proposition is proved.

In an exactly similar manner, changing X and Y into

$$\frac{1}{M}\frac{d}{dY}, \quad -\frac{1}{M}\frac{d}{dX},$$

it may be proved that

$$M^n f\left(\frac{d}{dy}, \quad -\frac{d}{dx}\right) u = F\left(\frac{d}{dY}, \quad -\frac{d}{dX}\right) U. \tag{2}$$

The results (1) and (2) may be applied to generate covariants and invariants, as we proceed to show.

Suppose $f(x, y)$ and u to be covariants of any third quantic v, where v may become identical with either as a particular case; also, denoting by $F_c(X, Y)$ and U_c the same covariants expressed in terms of the X, Y variables and the new coefficients of v after linear transformation, we have, by Prop. II., Art. 165, the identical equations

$$M^p F(X, Y) \equiv F_c(X, Y), \text{ and } M^q U \equiv U_c;$$

whence, substituting from these equations in (1),

$$M^r f\left(\frac{du}{dy}, \quad -\frac{du}{dx}\right) = F_c\left(\frac{dU_c}{dY}, \quad -\frac{dU_c}{dX}\right),$$

proving that $f\left(\frac{du}{dy}, \quad -\frac{du}{dx}\right)$ is a covariant of v.

And in a similar manner it is proved from (2) that

$$\left(\frac{d}{dy}, \ -\frac{d}{dx}\right) u$$

leads to an invariant or covariant of v, according as u is of the n^{th} or any higher order.

We add some applications of this method of forming invariants and covariants.

EXAMPLES.

1. If $\frac{d}{dy}$, $-\frac{d}{dx}$ be substituted for x and y in the quartic $(a, b, c, d, e)(x, y)^4 = U$, and the resulting operation performed on the quartic itself, show that the invariant I is obtained.

We find

$$(a, b, c, d, e)\left(\frac{d}{dy}, \ -\frac{d}{dx}\right)^4 U = 48\,(ae - 4bd + 3c^2).$$

2. Prove, by performing the same operation on H_x, the Hessian of the quartic (see Ex. 2, Art. 162), that the invariant J is obtained.

Here we find

$$(a, b, c, d, e)\left(\frac{d}{dy}, \ -\frac{d}{dx}\right)^4 H_x = 72\,(ace + 2bcd - ad^2 - eb^2 - c^3).$$

3. Prove that

$$(a, b, c, d)\left(\frac{d}{dy}, \ -\frac{d}{dx}\right)^3 G_x = -12\,(a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2),$$

where G_x is the cubic covariant of the cubic $(a, b, c, d)(x, y)^3$ (see Ex. 3, Art. 162).

4. Find the value of

$$(ac - b^2)\left(\frac{du}{dy}\right)^2 - (ad - bc)\frac{du}{dy}\frac{du}{dx} + (bd - c^2)\left(\frac{du}{dx}\right)^2,$$

where $u \equiv (a, b, c, d)(x, y)^3$.

Ans. $-9H_x^2$.

168. PROP II.—*If* $\phi(a_0, a_1, a_2, \ldots a_n)$ *be an invariant of the form* $(a_0, a_1, a_2, \ldots a_n)(x, y)^n$, *and* u *any quantic of the* n^{th} *or any higher degree,*

$$\phi\left(\frac{d^n u}{dx^n}, \ \frac{d^n u}{dx^{n-1}dy}, \ \frac{d^n u}{dx^{n-2}dy^2}, \ \ldots, \ \frac{d^n u}{dy^n}\right)$$

is an invariant or covariant of u. To prove this, let

$$x = \lambda X + \mu Y, \quad x' = \lambda X' + \mu Y',$$
$$y = \lambda' X + \mu' Y, \quad y' = \lambda' X' + \mu' Y';$$

and, transforming as in the last proposition,

$$x'\frac{d}{dx} + y'\frac{d}{dy} = X'\frac{d}{dX} + Y'\frac{d}{dY};$$

also transforming u, we have $U = u$; whence

$$\left(X'\frac{d}{dX} + Y'\frac{d}{dY}\right)^n U = \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^n u \qquad (1)$$

and writing this equation when expanded under the form

$$(D_0, D_1, D_2, \ldots D_n)(X', Y')^n = (d_0, d_1, d_2, \ldots d_n)(x', y')^n,$$

we have, from the definition of an invariant,

$$\phi(D_0, D_1, D_2, \ldots D_n) = M^q \phi(d_0, d_1, d_2, \ldots d_n),$$

showing that $\phi(d_0, d_1, d_2, \ldots d_n)$ is an invariant or covariant.

When x, y, and x', y' are transformed similarly, as in the present proposition, they are said to be *cogredient* variables.

The functions which occur in (1) are called *emanants;* the expression on the right-hand side of the equation being the n^{th} emanant of u.

EXAMPLES.

1. Let the quadratic

$$a_0x^2 + 2a_1xy + a_2y^2 \quad \text{become} \quad A_0X^2 + 2A_1XY + A_2Y^2.$$

We have then, as in Ex. 1, Art. 165,

$$A_0A_2 - A_1^2 = M^2(a_0a_2 - a_1^2).$$

Now since

$$X'^2\frac{d^2U}{dX^2} + 2X'Y'\frac{d^2U}{dX\,dY} + Y'^2\frac{d^2U}{dY^2} = x'^2\frac{d^2u}{dx^2} + 2x'y'\frac{d^2u}{dx\,dy} + y'^2\frac{d^2u}{dy^2},$$

it follows from the last result, considering X', Y' and x', y' as variables, that

$$\frac{d^2U}{dX^2}\frac{d^2U}{dY^2} - \left(\frac{d^2U}{dX\,dY}\right)^2 = M^2\left\{\frac{d^2u}{dx^2}\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2\right\}.$$

This gives an invariant of a quadratic, and a covariant (called the *Hessian*) of any higher quantic.

2. When u has the values

$$(a, b, c, d)(x, y)^3 \quad \text{and} \quad (a, b, c, d, e)(x, y)^4,$$

what covariants are derived by the process of the last example?

(Cf. Exs. 1, 2, Art. 162.)

Ans. (1). $(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$.

(2). $(ac - b^2)x^4 + 2(ad - bc)x^3y + (ae + 2bd - 3c^2)x^2y^2$
$+ 2(be - cd)xy^3 + (ce - d^2)y^4$.

169. Prop. III.—*If any invariant of the quantic in x, y,*

$$U + k\,(xy' - x'y)^n$$

be formed, the coefficients of the different powers of k, regarded as homogeneous functions of the variables x', y', are covariants of U.

For transforming U by linear transformation, let

$$(a_0,\ a_1,\ a_2,\ \ldots\ a_n)(x,\ y)^n = (A_0,\ A_1,\ A_2,\ \ldots\ A_n)(X,\ Y)^n;$$

also, if x, y and x', y' be cogredient variables,

$$xy' - x'y = M\,(XY' - X'Y).$$

Whence

$$(a_0,\ a_1,\ a_2,\ \ldots\ a_n)(x,\ y)^n + k\,(xy' - x'y)^n$$

becomes when transformed

$$(A_0,\ A_1,\ A_2,\ \ldots\ A_n)\,(X,\ Y)^n + kM^n\,(XY' - X'Y)^n\,;$$

and forming any invariant ϕ of both these forms, we have

$$(\phi,\ \phi_1,\ \phi_2,\ \ldots\ \phi_p)(1,\ k)^p = M^\kappa(\Phi,\ \Phi_1,\ \Phi_2,\ \ldots\ \Phi_p)(1,\ M^nk)^p,$$

proving that

$$\phi_r = M^q\,\Phi_r,$$

or that ϕ_r is a covariant.

When $(xy' - x'y)^n$ is replaced by $(b_0,\ b_1,\ b_2,\ \ldots\ b_n)\,(x,\ y)^n$, we have the following proposition which is established in a similar manner:—

If $\phi\,(a_0,\ a_1,\ a_2,\ \ldots\ a_n)$ be an invariant of $(a_0,\ a_1,\ a_2,\ \ldots\ a_n)(x,\ y)^n$, all the coefficients of k in

$$\phi\,(a_0 + kb_0,\quad a_1 + kb_1,\ \ldots\ a_n + kb_n)$$

are invariants of the system of two quantics

$$(a_0,\ a_1,\ a_2,\ \ldots\ a_n)(x,\ y)^n,\quad (b_0,\ b_1,\ b_2,\ \ldots\ b_n)\,(x,\ y)^n\,;$$

or, which is the same thing,

$$\left(b_0\frac{d}{da_0} + b_1\frac{d}{da_1} + \ldots + b_n\frac{d}{da_n}\right)^r \phi,\ \&c.,\ \&c.,$$

are invariants of the system.

If, further, ϕ be replaced by a covariant, we may in like manner generate new covariants; and these results hold for any number of quantics in any number of variables.

170. Prop. IV.—*If $\phi(x, y)$ and $\psi(x, y)$ are homogeneous quantics, the determinant*

$$\begin{vmatrix} \dfrac{d\phi}{dx} & \dfrac{d\phi}{dy} \\ \\ \dfrac{d\psi}{dx} & \dfrac{d\psi}{dy} \end{vmatrix}$$

is a covariant of these quantics.

For, transforming ϕ and ψ by the linear substitution

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

we have

$$\Phi(X, Y) = \phi(x, y), \quad \Psi(X, Y) = \psi(x, y),$$

giving

$$\frac{d\Phi}{dX} = \lambda \frac{d\phi}{dx} + \lambda' \frac{d\phi}{dy}, \quad \frac{d\Psi}{dX} = \lambda \frac{d\psi}{dx} + \lambda' \frac{d\psi}{dy},$$

$$\frac{d\Phi}{dY} = \mu \frac{d\phi}{dx} + \mu' \frac{d\phi}{dy}, \quad \frac{d\Psi}{dY} = \mu \frac{d\psi}{dx} + \mu' \frac{d\psi}{dy}.$$

Whence

$$\begin{vmatrix} \dfrac{d\Phi}{dX} & \dfrac{d\Phi}{dY} \\ \\ \dfrac{d\Psi}{dX} & \dfrac{d\Psi}{dY} \end{vmatrix} = \begin{vmatrix} \lambda \dfrac{d\phi}{dx} + \lambda' \dfrac{d\phi}{dy}, & \mu \dfrac{d\phi}{dx} + \mu' \dfrac{d\phi}{dy} \\ \\ \lambda \dfrac{d\psi}{dx} + \lambda' \dfrac{d\psi}{dy}, & \mu \dfrac{d\psi}{dx} + \mu' \dfrac{d\psi}{dy} \end{vmatrix},$$

which reduces to

$$M \left(\frac{d\phi}{dx} \frac{d\psi}{dy} - \frac{d\phi}{dy} \frac{d\psi}{dx} \right);$$

and the proposition is proved.

This covariant is called the *Jacobian* of ϕ and ψ, and is often written under the form $J(\phi, \psi)$. The Jacobian of n functions in n variables is a determinant of similar form, and can be shown to be a covariant by an exactly similar proof.

171. **Derivation of Invariants and Covariants by Differential Symbols.**—If x_1, y_1; x_2, y_2; x_3, y_3; ... x_n, y_n be a series of cogredient variables (such as, for example, the coordinates of n points), the functions $(x_1 y_2 - x_2 y_1), \ldots (x_p y_q - x_q y_p)$ are unaltered by linear transformation; and since $\dfrac{d}{dy_i}$, $-\dfrac{d}{dx_i}$

are transformed by the same linear transformation as x_i, y_i (see Art. 167), we derive a series of symbols of differentiation, which combined as above give the following :—

$$\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right), \ldots \left(\frac{d}{dx_p}\frac{d}{dy_q} - \frac{d}{dx_q}\frac{d}{dy_p}\right), \text{\&c.}$$

These symbols may be denoted simply by $(1, 2), \ldots (p, q)$, &c. ; and by their aid a complete calculus can be constructed for deriving and comparing invariants and covariants. For example, the Jacobian of ϕ, ψ may be written in the form

$$(1, 2)\,\phi_1\psi_2,$$

where $$\phi_1 = \phi(x_1, y_1), \quad \psi_2 = \psi(x_2, y_2),$$

the suffixes being omitted after the differentiation has been performed. Similarly, expanding the symbolic form $(1, 2)^2\phi_1\psi_2$, we obtain the covariant

$$\frac{d^2\phi}{dx^2}\frac{d^2\psi}{dy^2} - 2\frac{d^2\phi}{dxdy}\frac{d^2\psi}{dxdy} + \frac{d^2\phi}{dy^2}\frac{d^2\psi}{dx^2},$$

the distinction between the variables being removed after the differentiation has been performed.

In the investigation by this method of the invariants and covariants of a single quantic, the result is obtained under the symbolic form

$$(1, 2)^\alpha (2, 3)^\beta (3, 4)^\gamma \ldots (p, q)^\lambda\, U_1 U_2 U_3 \ldots U_p U_q,$$

where U_j, for example, is used to denote the quantic obtained by substituting x_j and y_j for x and y in U. If after this operation is performed x and y disappear, we have obtained an invariant ; and it is easy to see in this case that the figures $1, 2, 3, \ldots p, q$ must all occur exactly n times in terms such as $(i, j)^\kappa$. For example, the formula

$$(1, 2)^n\, U_1 U_2$$

gives a series of binary invariants for all *even* quantics, the order of the invariant in general being equal to the number of factors U_1, U_2, &c.

It should be noticed that this interchange of variables may be accomplished formally by means of a differential operator; for instance

$$\left(x\frac{d}{dx_j} + y\frac{d}{dy_j}\right)^n U_j = 1 . 2 . 3 \ldots n U, \text{ \&c. \&c.}$$

The method here explained of forming invariants and covariants is due to Prof. Cayley.

We now conclude this chapter with some examples selected to illustrate the foregoing theory. The student is referred for further information on this subject to Salmon's *Lessons Introductory to the Modern Higher Algebra*; and to Clebsch's *Theorie Der Binären Algebraischen Formen*, where a symbolic method is adopted throughout.

Examples.

1. The discriminant of any quantic is an invariant.

2. The resultant of two quantics is an invariant of the system.

3. From the definitions, Art. 159, prove that all the invariants of the quantic $U(xy' - x'y)$ are covariants of U, the variable being $x' : y'$.

Hence derive the covariants of a cubic from the invariants of a quartic expressed in terms of the roots.

4. If $I_1, I_2, I_3, \ldots I_n$ be the same invariant for each of the quantics $\frac{\phi(x)}{x - \alpha_1}, \frac{\phi(x)}{x - \alpha_2}, \frac{\phi(x)}{x - \alpha_3}, \ldots \frac{\phi(x)}{x - \alpha_n}$, of the order ϖ, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of $\phi(x) = 0$, prove that

$$\sum_{r=1}^{r=n} I_r (x - \alpha_r)^{\varpi}$$

is a covariant of $\phi(x)$.

For example, using J_1 to denote the J invariant composed of the four roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ (see Art. 160), with similar values for J_2, J_3, J_4, J_5, we have the following covariant of a quintic :—

$$J_1 (x - \alpha_1)^3 + J_2 (x - \alpha_2)^3 + J_3 (x - \alpha_3)^3 + J_4 (x - \alpha_4)^3 + J_5 (x - \alpha_5)^3.$$

5. If $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0;$$

and if

$$a_0^m \phi_1 \phi_2 \ldots \phi_m = F(a_0, a_1, a_2, \ldots a_n),$$

where $\phi_1, \phi_2, \ldots \phi_m$ are all the values of a rational and integral function of some or all the roots obtained by substitution, find the equation whose roots are the m values of $-\dfrac{\phi}{\delta\phi'}$, given $\delta^2\phi = 0$.

Ans. $F(U_0, U_1, U_2, \ldots U_n) = 0$.

6. Express the identical relation connecting three quadratics in terms of their invariants.

Let

$$U = a_1 x^2 + 2b_1 xy + c_1 y^2,$$

$$V = a_2 x^2 + 2b_2 xy + c_2 y^2,$$

$$W = a_3 x^2 + 2b_3 xy + c_3 y^2;$$

multiplying together the two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ y^2 & -xy & x^2 & 0 \end{vmatrix} \quad \begin{vmatrix} c_1 & -2b_1 & a_1 & 0 \\ c_2 & -2b_2 & a_2 & 0 \\ c_3 & -2b_3 & a_3 & 0 \\ x^2 & 2xy & y^2 & 0 \end{vmatrix},$$

we have

$$4\begin{vmatrix} I_{11} & I_{12} & I_{13} & U \\ I_{12} & I_{22} & I_{23} & V \\ I_{13} & I_{23} & I_{33} & W \\ U & V & W & 0 \end{vmatrix} \equiv 0, \quad \text{where} \quad 2I_{pq} = a_p c_q + a_q c_p - 2b_p b_q.$$

Expanding this determinant we have

$$(I_{22} I_{33} - I_{23}^2) U^2 + (I_{33} I_{11} - I_{31}^2) V^2 + (I_{11} I_{22} - I_{12}^2) W^2 + 2(I_{31} I_{12} - I_{11} I_{23}) VW$$
$$+ 2(I_{12} I_{23} - I_{22} I_{31}) WU + 2(I_{23} I_{31} - I_{33} I_{12}) UV \equiv 0. \qquad (1)$$

There are two particular cases worth noticing :—

(1). *When the three quadratics are mutually harmonic.*—In this case $I_{23} = 0$, $I_{31} = 0$, $I_{12} = 0$; and the identical equation assumes the following simple form :—

$$\left(\frac{U}{\sqrt{I_{11}}}\right)^2 + \left(\frac{V}{\sqrt{I_{22}}}\right)^2 + \left(\frac{W}{\sqrt{I_{33}}}\right)^2 \equiv 0.$$

(2). *When one of the quadratics $W = 0$ determines the foci of the involution of the points given by the other two, $U = 0$, and $V = 0$.*—In this case $I_{13} = 0$, and $I_{23} = 0$; and making this reduction in the general equation (1), we have

$$(I_{12}^2 - I_{11} I_{22}) W^2 = I_{33}(I_{22} U^2 - 2I_{12} UV + I_{11} V^2);$$

but from the equations $I_{13} = 0$, and $I_{23} = 0$, we find

$$a_3 = \kappa\,(a_1 b_2), \quad -2b_3 = \kappa\,(c_1 a_2), \quad c_3 = \kappa\,(b_1 c_2);$$

whence

$$4\,(a_3 c_3 - b_3^2) = \kappa^2\{4\,(a_1 b_2)(b_1 c_2) - (c_1 a_2)^2\},$$

or

$$I_{33} = \kappa^2\{I_{11} I_{22} - I_{12}^2\},$$

and reducing, when $\kappa = 1$, or $W \equiv J(U, V)$,

$$-\{J(U, V)\}^2 = I_{22} U^2 - 2I_{12} UV + I_{11} V^2.$$

7. Determine the invariants of the quartic

$$\lambda_1 (x - \alpha_1)^4 + \lambda_2 (x - \alpha_2)^4 + \ldots + \lambda_n (x - \alpha_n)^4.$$

Ans. $I = \Sigma\lambda_1 \lambda_2 (\alpha_1 - \alpha_2)^4$, $J = \Sigma\lambda_1 \lambda_2 \lambda_3 \nabla(\alpha_1, \alpha_2, \alpha_3)$,

where $\nabla(\alpha_1, \alpha_2, \ldots \alpha_r)$ represents the product of the squared differences of $\alpha_1, \alpha_2, \ldots \alpha_r$.

8. Prove that the condition that four roots of an equation of the n^{th} degree should determine on a right line a harmonic system of points may be expressed by equating to zero an invariant of the degree $\frac{1}{2}(n-1)(n-2)(n-3)$.

9. If $\phi(a_0, a_1, \ldots a_n)$ be any seminvariant of the quantic $(a_0, a_1, \ldots a_n)(x, 1)^n$; prove that $\dfrac{d\phi}{da_n}$ is also a seminvariant.

10. Prove that the seminvariants

$$a_0 a_2 - a_1^2, \quad a_0 a_4 - 4a_1 a_3 + 3a_2^2, \quad a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$

of the quantic $(a_0, a_1, a_2, \ldots a_n)(x, y)^n$ give rise to covariants of the degrees

$$2n - 4, \qquad 2n - 8, \quad 3n - 6.$$

11. Prove that the coefficient of the penultimate term in the equation of the squares of the differences of any quantic leads to a covariant of that quantic of the fourth degree in the variables.

12. Prove that the product of two covariants of the same quantic whose sources are ϕ and ψ may be written under the form

$$\phi\psi + xD\,(\phi\psi) + \frac{x^2}{1\,.\,2}\,D^2\,(\phi\psi) + \&c. \ldots$$

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13. Prove in particular that the m^{th} power of the quantic

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n$$

may be represented by

$$a_n^m + xD\,(a_n^m) + \frac{x^2}{1\,.\,2}\,D^2\,(a_n^m) + \frac{x^3}{1\,.\,2\,.\,3}\,D^3\,(a_n^m) + \&c.$$

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14. Prove from both definitions of a covariant that any covariant of a covariant is a covariant of the original quantic or quantics.

15. If $\alpha_1, \alpha_2, \ldots \alpha_m$, and $\beta_1, \beta_2, \ldots \beta_n$ be the roots of the equations

$$U \equiv (a_0, a_1, a_2, \ldots a_m)(x, 1)^m = 0, \quad \text{and} \quad V \equiv (b_0, b_1, b_2, \ldots b_n)(x, 1)^n = 0;$$

from the simplest function of the differences of their roots, viz., $\Sigma(\alpha_p - \beta_q)$, it is required to derive a covariant of the system U and V.

This question will be solved if we express

$$UV \sum \frac{\alpha_p - \beta_q}{(x - \alpha_p)(x - \beta_q)}$$

in terms of the coefficients of U and V.

For this purpose we have

$$\sum \frac{\alpha_p - \beta_q}{(x - \alpha_p)(x - \beta_q)} = \sum \frac{\alpha}{x - \alpha} \sum \frac{1}{x - \beta} - \sum \frac{\beta}{x - \beta} \sum \frac{1}{x - \alpha},$$

and if U and V be written as homogeneous functions of x and y,

$$\sum \frac{1}{x - \alpha y} = \frac{d \log U}{dx}, \quad \sum \frac{\alpha}{x - \alpha y} = -\frac{d \log U}{dy}, \text{ \&c.}$$

Whence, substituting these values in the last equation, we have

$$UV \sum \frac{\alpha_p - \beta_q}{(x - \alpha_p y)(x - \beta_q y)} = \frac{dU}{dx}\frac{dV}{dy} - \frac{dU}{dy}\frac{dV}{dx};$$

which is the Jacobian of U and V. It should be noticed also that the leading coefficient of $J(U, V)$ is $a_0 b_1 - a_1 b_0$.

16. Prove that the common factors of two quantics are double factors of their Jacobian $J(U, V)$, when the quantics are of the same degree n.

Let $U \equiv P\phi$, $V \equiv P\psi$, where $P \equiv lx + my$. Forming $J(U, V)$, we find part of it divisible by P^2, and the part which apparently has only P as a factor may be written as follows (using Euler's theorem of homogeneous functions, and omitting a numerical factor):—

$$\left(x\frac{d\phi}{dx} + y\frac{d\phi}{dy}\right)\left(l\frac{d\psi}{dy} - m\frac{d\psi}{dx}\right) + \left(x\frac{d\psi}{dx} + y\frac{d\psi}{dy}\right)\left(m\frac{d\phi}{dx} - l\frac{d\phi}{dy}\right),$$

and this is identical with $(lx + my)\,J(\phi, \psi)$.

17. Prove that the $2(n-1)$ double factors of $\lambda U + \mu V$, obtained by varying λ and μ, are the factors of $J(U, V)$, where U and V are both of the n^{th} degree.

18. Find the resultant of two cubics U and V by eliminating dialytically between

$$U = 0, \quad V = 0, \quad \frac{dJ(U, V)}{dx} = 0, \quad \frac{dJ(U, V)}{dy} = 0.$$

CHAPTER XVI.

COVARIANTS AND INVARIANTS OF THE QUADRATIC, CUBIC, AND QUARTIC.

172. **The Quadratic.**—*The quadratic has only one invariant, and no covariant other than the quadratic itself.*

For, if α and β be the roots of the quadratic equation

$$U \equiv ax^2 + 2bx + c = 0,$$

the only functions of their difference which can lead to an invariant or covariant are powers of $\alpha - \beta$ of the type $(\alpha - \beta)^{2p}$; the odd powers of $\alpha - \beta$ not being expressible by the coefficients in a rational form. Whence, expressing

$$U^{2p}\left(\frac{1}{\alpha - x} - \frac{1}{\beta - x}\right)^{2p}$$

by the coefficients, we conclude that the quadratic has only the one distinct invariant $ac - b^2$, and no covariant distinct from U itself.

173. **The Cubic and its Covariants.**—In the present Article the covariants of the cubic will be discussed as examples of the principles already explained, and in the following Article the definite number of covariants and invariants will be determined.

In the case of the cubic a covariant is obtained from a function of the differences of the roots most simply by substituting

$$\beta\gamma + \alpha x,\ \gamma\alpha + \beta x,\ \alpha\beta + \gamma x \text{ for } -\alpha,\ -\beta,\ -\gamma,$$

and thus avoiding fractions; for, transforming $\alpha - \beta$, we have

$$\frac{1}{\alpha - x} - \frac{1}{\beta - x} = \frac{-(\beta\gamma + \alpha x) + (\gamma\alpha + \beta x)}{(x - \alpha)(x - \beta)(x - \gamma)};$$

and when fractions are removed we arrive at the above transformation (the order being equal to the weight in the case of either function of the differences H or G). This mode of transforming functions of the differences will now be applied to the covariants of the cubic.

(1). *The Quadratic Covariant, or Hessian,* H_x.

Transforming both sides of the equation

$$a_0^2\,(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) = 9\,(a_1^2 - a_0a_2),$$

we have

$$a_0^2\,\{(\alpha + \omega\beta + \omega^2\gamma)\,x + \beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta\}$$

$$\times\,\{(\alpha + \omega^2\beta + \omega\gamma)\,x + \beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta\} = 9\,(U_2^2 - U_3U_1)\,;$$

thus showing that

$$Lx + L_1 \text{ and } Mx + M_1 \qquad \text{(See Art. 59.)}$$

are the factors of

$$H_x = (a_0a_2 - a_1^2)\,x^2 + (a_0a_3 - a_1a_2)\,x + (a_1a_3 - a_2^2),$$

where

$$L_1 = \beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta, \quad M_1 = \beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta.$$

From the form of the Hessian in terms of the roots in Art. 160, or from the relations of Art. 43, we conclude that *when a cubic is a perfect cube, each of the coefficients of the Hessian vanishes identically.*

(2). *The Cubic Covariant,* G_x.

We have, as in Art. 59,

$$a_0^3\{(\alpha + \omega\beta + \omega^2\gamma)^3 + (\alpha + \omega^2\beta + \omega\gamma)^3\} = -27\,(a_0^2a_3 + 2a_1^3 - 3a_0a_1a_2).$$

Transforming both sides of this equation as before, we find

$$a_0^3\{(Lx + L_1)^3 + (Mx + M_1)^3\} = -27\,(U^2U_0 + 2U_2^3 - 3U_1U_2U)$$

$$= 27\,G_x,$$

where G_x denotes the covariant formed from the function of differences G; and operating as in Art. 162 on the source derived from G (the sign being changed in order that G may be the leading coefficient), we easily obtain (see Ex. 3, Art. 162)

$$G_x = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)\,x^3 + 3\,(a_0a_1a_3 + a_1^2a_2 - 2a_0a_2^2)\,x^2$$

$$-\,(a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3) \quad - 3\,(a_3a_2a_0 + a_2^2a_1 - 2a_3a_1^2)\,x.$$

Resolving $(Lx+L_1)^3+(Mx+M_1)^3$, we may obtain the factors of G_x; or, more simply, since the factors of G are $\beta+\gamma-2\alpha$, $\gamma+\alpha-2\beta$, $\alpha+\beta-2\gamma$, the factors of G_x are

$$\frac{1}{\beta-x}+\frac{1}{\gamma-x}-\frac{2}{\alpha-x},\ \frac{1}{\gamma-x}+\frac{1}{\alpha-x}-\frac{2}{\beta-x},\ \frac{1}{\alpha-x}+\frac{1}{\beta-x}-\frac{2}{\gamma-x},$$

when fractions are removed.

We have obviously the following geometrical interpretation of the equation $G_x=0$:—If three points A, B, C determined by the equation $U=0$ be taken on a right line; and three points A', B', C', such that A' is the harmonic conjugate of A with regard to B and C; B' of B with regard to C and A; and C' of C with regard to A and B; the points A', B', C' are determined by the equation $G_x=0$. (Compare Ex. 13, p. 88.)

(3). *Expression of the Cubic as the difference of two cubes.*

This can be effected, by means of the factors of the Hessian, as follows:—

$$(Lx+L_1)^3-(Mx+M_1)^3=27U\frac{\sqrt{\Delta}}{a_0^3}.$$

For, as in Ex. 6, p. 116, we have

$$L^3-M^3=\sqrt{-27}\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Transforming this equation as before, the first side becomes

$$(Lx+L_1)^3-(Mx+M_1)^3,$$

and the second side

$$\sqrt{-27}\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

Substituting from previous equations, we have

$$(Lx+L_1)^3-(Mx+M_1)^3=27\frac{U}{a_0^4}\sqrt{G^2+4H^3}=27\frac{U\sqrt{\Delta}}{a_0^3}.$$

(4). *Relation between the Cubic and its Covariants.*

The following relation exists:—

$$G_x^2+4H_x^3=\Delta U^2.$$

For, from Ex. 6, p. 116,

$$a_0^6(\beta-\gamma)^2(\gamma-a)^2(a-\beta)^2=-27(G^2+4H^3)=-27a_0^2\Delta,$$

and transforming this equation as before,

$$a_0^6(\beta-\gamma)^2(\gamma-a)^2(a-\beta)^2(x-a)^2(x-\beta)^2(x-\gamma)^2=-27(G_x^2+4H_x^3);$$

whence $$\Delta U^2=G_x^2+4H_x^3.$$

(5). *Solution of the Cubic.*

The expression

$$(U\sqrt{\Delta}+G_x)^{\frac{1}{3}}+(U\sqrt{\Delta}-G_x)^{\frac{1}{3}}$$

is a linear factor of U.

For, from the relations in (2) and (3), we have

$$2a_0^3(Lx+L_1)^3=27(U\sqrt{\Delta}-G_x),$$

$$-2a_0^3(Mx+M_1)^3=27(U\sqrt{\Delta}+G_x);$$

and since

$$(Lx+L_1)-(Mx+M_1)$$

is a factor of U, the proposition follows.

This form of solution of the cubic is due to Prof. Cayley.

174. **Number of Covariants and Invariants of the Cubic.**—The following method of determining the number of covariants and invariants of the cubic is similar to that employed by Professor Cayley for the same purpose :—

The cubic has only two covariants, their leading terms being H and G; and only one invariant, viz. the discriminant Δ, where

$$a^2\Delta=G^2+4H^3, \text{ or } \Delta=a^2d^2+4ac^3-6abcd+4db^3-3b^2c^2.$$

To prove this, let $\phi(a,\beta,\gamma)$ be any integral symmetric function of the *differences* of the roots (of order ϖ), expressible by the coefficients in a rational form.

We have then (Art. 36),

$$a^r\phi(a,\beta,\gamma)=F(a,H,G) \qquad (1)$$

(where r remains to be determined); and, in the first place, if ϕ be an even function of the roots, G can enter this equation in even powers only, since H is an even function of the roots.

Eliminating the even powers of G by means of the relation

$$G^2 + 4H^3 = a^2\Delta,$$

we show therefore that in the case of an even function of the roots equation (1) takes the form

$$a^r \phi(\alpha, \beta, \gamma) = F(a, H, \Delta),$$

which may be written

$$a^{\varpi} \phi(\alpha, \beta, \gamma) = F_0(a, H, \Delta) + \Sigma \frac{F_p(H, \Delta)}{a^p}, \qquad (2)$$

where ϖ is the order of $\phi(\alpha, \beta, \gamma)$, and F_0 an integral function.

It is now necessary to prove the following Lemma :—

No function of H and Δ exists which is divisible by a.

For, suppose $F_p(H, \Delta)$ to be divisible by a; then making a vanish, we have

$$F_p(H', \Delta') \equiv 0,$$

where $H' = -b^2$, $\Delta' = 4db^3 - 3b^2c^2$, the values of H and Δ when a vanishes. This equation is plainly impossible; for, eliminating b by means of the equation $H' = -b^2$, c and d remain in the equation connecting H' and Δ'.

From this it follows that equation (2) must assume the form

$$a^{\varpi} \phi(\alpha, \beta, \gamma) = F_0(a, H, \Delta),$$

for the first side of this equation is expressible as an integral function of the coefficients; therefore so must the second side also, and consequently the fractional part must disappear.

Now, to extend this result to odd functions of the roots, we have only to multiply the first side of the equation by

$$a^3(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta),$$

and the second side by $27G$, for G must be a factor of every odd function, since H is even.

We are now in a position to prove the original proposition as to the number of invariants and covariants. For since $a^{\varpi}\phi$ is of the form

$$GF(a, H, \Delta), \text{ or } F(a, H, \Delta),$$

according as ϕ is an odd or even function of the roots, it follows

in the first place that there cannot be an invariant of an odd degree in the roots, since $GF(a, H, \Delta)$ does not remain the same function when a, b, c, d are changed into d, c, b, a, respectively; and the only invariant of an even degree must be a power of Δ, since if $F(a, H, \Delta)$ contained a or H besides Δ, it could not remain the same function when the coefficients are similarly interchanged.

Again, the cubic has only two distinct covariants; for it has been proved that every seminvariant $a^\varpi\phi$ is of one of the forms

$$F(a, H, \Delta), \quad \text{or} \quad GF(a, H, \Delta);$$

and therefore the corresponding covariant, formed from the seminvariant as leading term, must be expressible as

$$F(U, H_x, \Delta), \quad \text{or} \quad G_xF(U, H_x, \Delta);$$

that is, every covariant is expressible in a rational and integral form in terms of H_x and G_x, along with U and Δ; or in other words, there are only two distinct covariants.

175. **The Quartic. Its Covariants and Invariants.**—We have shown already that the quartic has two invariants, I and J (see Art. 160). From the functions H and G of the differences of the roots we can derive two covariants H_x and G_x, whose leading coefficients are H and G; for from the relation

$$a_0^2\,\Sigma\,(\alpha-\beta)^2 = 48\,(a_0a_2 - a_1^2)$$

we derive, by the process of Art. 160,

$$a_0^2\,\Sigma\,(\alpha-\beta)^2(x-\gamma)^2(x-\delta)^2 = 48\,(UU_2 - U_3^2);$$

and, expanding $UU_2 - U_3^2$, we have

$$H_x \equiv (a_0a_2 - a_1^2)\,x^4 + 2\,(a_0a_3 - a_1a_2)\,x^3 + (a_0a_4 + 2a_1a_3 - 3a_2^2)\,x^2$$
$$+ 2\,(a_1a_4 - a_2a_3)\,x + (a_2a_4 - a_3^2).$$

In a similar manner, since

$$G = a_0^2a_3 + 2a_1^3 - 3a_0a_1a_2,$$

we obtain the covariant

$$-G_x \equiv U^2U_1 + 2U_3^3 - 3UU_3U_2,$$

which reduces to the sixth degree; and if it be written as follows:—

$$G_x \equiv A_0x^6 + A_1x^5 + A_2x^4 + A_3x^3 + A_4x^2 + A_5x + A_6,$$

we find, by expanding the above, or more simply, by forming the source A_6, and performing the successive operations of Art. 162, the following values of the coefficients:—

$A_6 = -a_4^2a_1 + 3a_4a_3a_2 - 2a_3^3$, $\quad A_5 = -a_4^2a_0 - 2a_4a_3a_1 - 6a_3^2a_2 + 9a_4a_2^2$,

$A_4 = -5a_4a_3a_0 - 10a_3^2a_1 + 15a_4a_2a_1$, $\quad A_3 = -10a_0a_3^2 + 10a_1^2a_4$,

$A_2 = 5a_0a_1a_4 + 10a_1^2a_3 - 15a_0a_2a_3$, $\quad A_1 = a_0^2a_4 + 2a_0a_1a_3 + 6a_1^2a_2 - 9a_0a_2^2$,

$A_0 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$.

Here it will be observed that, when A_3 is determined, A_2, A_1, and A_0 may be obtained from A_4, A_5, and A_6 by changing the suffixes into their complementary values, and altering the sign of the whole, in accordance with what was proved in Art. 161.

We proceed in the following Articles to discuss the leading properties of these two covariants of the quartic.

176. **Quadratic Factors of the Sextic Covariant.***— As the quadratic factors of G_x enter prominently into the following discussion, we proceed in the first place to find expressions for those factors in terms of the roots of the quartic, and to deduce their principal properties.

Since the factors of G, expressed in terms of α, β, γ, δ, are

$$\beta + \gamma - \alpha - \delta, \quad \gamma + \alpha - \beta - \delta, \quad \alpha + \beta - \gamma - \delta,$$

the factors of G_x are obtained from these by substituting $\dfrac{1}{x-\alpha}$, $\dfrac{1}{x-\beta}$, $\dfrac{1}{x-\gamma}$, $\dfrac{1}{x-\delta}$, for α, β, γ, δ, respectively, and multiplying each factor by $\dfrac{U}{a}$ to remove fractions.

* See a Paper by Prof. Ball, *Quarterly Journal of Mathematics*, vol. vii., p. 368, containing a full and valuable discussion of the various solutions of the biquadratic.

Whence, denoting these factors by u, v, w, we have

$$au = U\left(\frac{1}{x-\beta} + \frac{1}{x-\gamma} - \frac{1}{x-a} - \frac{1}{x-\delta}\right),$$

$$av = U\left(\frac{1}{x-\gamma} + \frac{1}{x-a} - \frac{1}{x-\beta} - \frac{1}{x-\delta}\right), \tag{1}$$

$$aw = U\left(\frac{1}{x-a} + \frac{1}{x-\beta} - \frac{1}{x-\gamma} - \frac{1}{x-\delta}\right),$$

which values of u, v, w, arranged in powers of x, are

$$u = (\beta + \gamma - a - \delta)\, x^2 - 2\,(\beta\gamma - a\delta)\, x + \beta\gamma\,(a + \delta) - a\delta\,(\beta + \gamma),$$

$$v = (\gamma + a - \beta - \delta)\, x^2 - 2\,(\gamma a - \beta\delta)\, x + \gamma a\,(\beta + \delta) - \beta\delta\,(\gamma + a), \tag{2}$$

$$w = (a + \beta - \gamma - \delta)\, x^2 - 2\,(a\beta - \gamma\delta)\, x + a\beta\,(\gamma + \delta) - \gamma\delta\,(a + \beta);$$

and, consequently, $32G_x = a^3uvw$.

From equations (1) we easily find

$$v = (a-\delta)(x-\beta)(x-\gamma) - (\beta-\gamma)(x-a)\,(x-\delta),$$

$$w = (a-\delta)(x-\beta)(x-\gamma) + (\beta-\gamma)(x-a)(x-\delta);$$

and from these and similar equations we have

$$\frac{v^2 - w^2}{\mu - \nu} = \frac{w^2 - u^2}{\nu - \lambda} = \frac{u^2 - v^2}{\lambda - \mu} = 4\,\frac{U}{a}, \tag{3}$$

where λ, μ, ν have the usual meaning (Ex. 17, Art. 27); and consequently,

$$(\mu - \nu)u^2 + (\nu - \lambda)\, v^2 + (\lambda - \mu)\, w^2 \equiv 0;$$

whence

$$-(\mu - \nu)\, u^2 \equiv (w\sqrt{\lambda - \mu} + v\sqrt{\lambda - \nu})(w\sqrt{\lambda - \mu} - v\sqrt{\lambda - \nu}).$$

Since, as this identical equation shows, the factors on the second side are both perfect squares, we may assume

$$w\sqrt{\lambda - \mu} + v\sqrt{\lambda - \nu} \equiv 2u_1^2,$$

$$w\sqrt{\lambda - \mu} - v\sqrt{\lambda - \nu} \equiv 2u_2^2;$$

we have, therefore,

$$w\sqrt{\lambda - \mu} = u_1^2 + u_2^2,$$

$$v\sqrt{\lambda - \nu} = u_1^2 - u_2^2,$$

$$u\sqrt{\nu - \mu} = 2u_1u_2;$$

from which values we conclude that u, v, w, *the quadratic factors of* G_x, *are mutually harmonic.*

For the geometrical interpretation of the equation $G_x = 0$ see Art. 65.

177. **Expression of the Hessian by the Quadratic Factors of G_x.**—Since

$$-48\frac{H_x}{a^2} = \Sigma(\alpha - \beta)^2(x - \gamma)^2(x - \delta)^2;$$

combining the terms in pairs, and noticing that

$$\Sigma(\beta - \gamma)(\alpha - \delta)U \equiv 0,$$

$$\Sigma(\alpha - \beta)^2(x - \gamma)^2(x - \delta)^2$$
$$= \Sigma\{(\beta - \gamma)(x - \alpha)(x - \delta) + (\alpha - \delta)(x - \beta)(x - \gamma)\}^2,$$

the quantities between brackets being u, v, w, we have

$$-48\frac{H_x}{a^2} = u^2 + v^2 + w^2,$$

which is the required expression for H_x.

178. **Expression of the Quartic itself by the Quadratic Factors of G_x.**—From equations (3) a symmetrical value may be obtained for U; for, substituting in those equations in place of λ, μ, ν their values in terms of the roots ρ_1, ρ_2, ρ_3 of the equation $4\rho^3 - I\rho + J = 0$, we find

$$a^2(v^2 - w^2) = 16(\rho_2 - \rho_3)U, \quad a^2(w^2 - u^2) = 16(\rho_3 - \rho_1)U,$$
$$a^2(u^2 - v^2) = 16(\rho_1 - \rho_2)U,$$

from which equations, by means of the value of H_x in the preceding Article, we obtain

$$(au)^2 = 16(\rho_1U - H_x), \quad (av)^2 = 16(\rho_2U - H_x), \qquad (4)$$
$$(aw)^2 = 16(\rho_3U - H_x).$$

We now make the substitutions

$$u^2 = \Delta_1 X^2, \quad v^2 = \Delta_2 Y^2, \quad w^2 = \Delta_3 Z^2,$$

where Δ_1, Δ_2, Δ_3 are the discriminants of u, v, w; thus replacing u, v, w by three quadratics X, Y, Z whose discriminants are each equal to unity. By means of this transformation the forms of the quadratics are further fixed, and the identical relation connecting their squares (see (1), Ex. 6, p. 389) is expressed in its simplest form. Calculating the discriminants, we find

$$\Delta_1 = (\beta + \gamma - \alpha - \delta)\{\beta\gamma(\alpha + \delta) - \alpha\delta(\beta + \gamma)\} - (\beta\gamma - \alpha\delta)^2,$$

with similar values of Δ_2 and Δ_3; whence we have

$$\Delta_1 = -(\lambda - \mu)(\lambda - \nu), \quad \Delta_2 = -(\mu - \nu)(\mu - \lambda), \quad \Delta_3 = -(\nu - \lambda)(\nu - \mu).$$

Making these substitutions, the preceding equations become

$$\begin{aligned} (\rho_1 - \rho_2)(\rho_1 - \rho_3) X^2 &= H_x - \rho_1 U, \\ (\rho_2 - \rho_3)(\rho_2 - \rho_1) Y^2 &= H_x - \rho_2 U, \\ (\rho_3 - \rho_1)(\rho_3 - \rho_2) Z^2 &= H_x - \rho_3 U; \end{aligned} \tag{5}$$

from which are easily deduced the following values of U and H_x, and the identical equation connecting X, Y, Z:—

$$\begin{aligned} H_x &= \rho_1^2 X^2 + \rho_2^2 Y^2 + \rho_3^2 Z^2, \\ -U &= \rho_1 X^2 + \rho_2 Y^2 + \rho_3 Z^2, \\ 0 &= X^2 + Y^2 + Z^2; \end{aligned} \tag{6}$$

where, as has been proved, X, Y, Z are three mutually harmonic quadratics whose discriminants are reduced to unity in each case. The value of G_x may be expressed in terms of X, Y, Z as follows. Since $32 G_x = a^3 uvw$, and

$$u^2v^2w^2 = (\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2 X^2 Y^2 Z^2 = \frac{256}{a^6}(I^3 - 27J^2) X^2 Y^2 Z^2,$$

we find

$$G_x = \tfrac{1}{2}\sqrt{I^3 - 27J^2} \,.\, XYZ.$$

179. **Resolution of the Quartic.**—From the equations

$$-U = \rho_1 X^2 + \rho_2 Y^2 + \rho_3 Z^2,$$

$$0 = X^2 + Y^2 + Z^2,$$

we find

$$U = (\rho_1 - \rho_2) Y^2 + (\rho_1 - \rho_3) Z^2, \quad U = (\rho_2 - \rho_3) Z^2 + (\rho_2 - \rho_1) X^2,$$

$$U = (\rho_3 - \rho_1) X^2 + (\rho_3 - \rho_2) Y^2,$$

where X^2, Y^2, Z^2 have the values determined by equations (5); and breaking up these values of U into their factors, we have three ways of resolving U depending on the solution of the equation

$$4\rho^3 - I\rho + J = 0.$$

The resolution of the quartic has been presented by Professor Cayley in a symmetrical form which may be easily derived from the expressions already given for U and H_x. For, since in general

$$l(a_1x^2 + 2b_1xy + c_1y^2) + m(a_2x^2 + 2b_2xy + c_2y^2) + n(a_3x^2 + 2b_3xy + c_3y^2)$$

is a perfect square when

$$\Sigma l^2 (a_1c_1 - b_1^2) + \Sigma mn (a_2c_3 + a_3c_2 - 2b_2b_3) = 0,$$

$$lX + mY + nZ \text{ is a perfect square when } l^2 + m^2 + n^2 = 0,$$

X, Y, Z being mutually harmonic, and the discriminants each reduced to unity.

The resolution of U is therefore reduced to finding values of l, m, n such that $lX + mY + nZ$, or

$$l\sqrt{\rho_2 - \rho_3}\sqrt{H_x - \rho_1 U} + m\sqrt{\rho_3 - \rho_1}\sqrt{H_x - \rho_2 U}$$
$$+ n\sqrt{\rho_1 - \rho_2}\sqrt{H_x - \rho_3 U},$$

being a perfect square, may vanish when U vanishes; or in fact to satisfy the two equations

$$l\sqrt{\rho_2 - \rho_3} + m\sqrt{\rho_3 - \rho_1} + n\sqrt{\rho_1 - \rho_2} = 0, \quad l^2 + m^2 + n^2 = 0.$$

These equations are plainly satisfied if

$$\frac{l}{\sqrt{\rho_2 - \rho_3}} = \frac{m}{\sqrt{\rho_3 - \rho_1}} = \frac{n}{\sqrt{\rho_1 - \rho_2}};$$

whence, finally, the squares of the four linear factors of U are

$$(\rho_2 - \rho_3)\sqrt{H_x - \rho_1 U} \pm (\rho_3 - \rho_1)\sqrt{H_x - \rho_2 U} \pm (\rho_1 - \rho_2)\sqrt{H_x - \rho_3 U},$$

which expression when rationalised becomes ΔU^2.

If it be required to resolve the quartic $\kappa U - \lambda H_x$, it appears in a similar manner that

$$l\sqrt{\rho_2 - \rho_3}\sqrt{H_x - \rho_1 U} + m\sqrt{\rho_3 - \rho_1}\sqrt{H_x - \rho_2 U} \\ + n\sqrt{\rho_1 - \rho_2}\sqrt{H_x - \rho_3 U},$$

being a perfect square, must vanish when $\kappa U - \lambda H_x$ vanishes; or, values of l, m, n must be determined so as to satisfy the equations

$$l^2 + m^2 + n^2 = 0,$$

$$l\sqrt{(\rho_2 - \rho_3)(\kappa - \rho_1\lambda)} + m\sqrt{(\rho_3 - \rho_1)(\kappa - \rho_2\lambda)} + n\sqrt{(\rho_1 - \rho_2)(\kappa - \rho_3\lambda)} = 0.$$

These equations are plainly satisfied if

$$\frac{l}{\sqrt{(\rho_2 - \rho_3)(\kappa - \rho_1\lambda)}} = \frac{m}{\sqrt{(\rho_3 - \rho_1)(\kappa - \rho_2\lambda)}} = \frac{n}{\sqrt{(\rho_1 - \rho_2)(\kappa - \rho_3\lambda)}};$$

whence

$$(\rho_2 - \rho_3)\sqrt{\kappa - \rho_1\lambda}\sqrt{H_x - \rho_1 U} + (\rho_3 - \rho_1)\sqrt{\kappa - \rho_2\lambda}\sqrt{H_x - \rho_2 U} \\ + (\rho_1 - \rho_2)\sqrt{\kappa - \rho_3\lambda}\sqrt{H_x - \rho_3 U}$$

is the square of a linear factor of $\kappa U - \lambda H_x$.

180. **The Invariants and Covariants of** $\kappa U - \lambda H_x$. —Employing the equations (6) of Art. 178, and denoting $X^2 + Y^2 + Z^2$ by V, we may, by adding $-\dfrac{\lambda I}{6} V$ to $\lambda H_x - \kappa U$, reduce it to the form $R_1 X^2 + R_2 Y^2 + R_3 Z^2$, where $R_1 + R_2 + R_3 = 0$. When this is done we have the following reduced values of R_1, R_2, R_3:—

$$3R_1 = \kappa(2\rho_1 - \rho_2 - \rho_3) + \lambda(2\rho_2\rho_3 - \rho_3\rho_1 - \rho_1\rho_2),$$

$$3R_2 = \kappa(2\rho_2 - \rho_3 - \rho_1) + \lambda(2\rho_3\rho_1 - \rho_1\rho_2 - \rho_2\rho_3),$$

$$3R_3 = \kappa(2\rho_3 - \rho_1 - \rho_2) + \lambda(2\rho_1\rho_2 - \rho_2\rho_3 - \rho_3\rho_1).$$

On account of the similarity of the forms

$$\rho_1 X^2 + \rho_2 Y^2 + \rho_3 Z^2 \text{ and } R_1 X^2 + R_2 Y^2 + R_3 Z^2,$$

which are of the same type, we calculate the invariants and covariants of $\kappa U - \lambda H_x$ by simply changing ρ_1, ρ_2, ρ_3 into R_1, R_2, R_3 in the expressions for the invariants and covariants of U.

Therefore, since

$$I = \tfrac{2}{3}\{(\rho_2 - \rho_3)^2 + (\rho_3 - \rho_1)^2 + (\rho_1 - \rho_2)^2\}, \quad J = -4\rho_1\rho_2\rho_3,$$

and

$$R_2 - R_3 = (\rho_2 - \rho_3)(\kappa - \lambda\rho_1), \quad R_3 - R_1 = (\rho_3 - \rho_1)(\kappa - \lambda\rho_2),$$

$$R_1 - R_2 = (\rho_1 - \rho_2)(\kappa - \lambda\rho_3),$$

we find the following values for the invariants of $\kappa U - \lambda H_x$:—

$$I_{(\kappa,\lambda)} = I\kappa^2 - 3J\kappa\lambda + \frac{I^2}{12}\lambda^2,$$

$$J_{(\kappa,\lambda)} = J\kappa^3 - \frac{I^2}{6}\kappa^2\lambda + \frac{IJ}{4}\kappa\lambda^2 - \frac{54J^2 - I^3}{216}\lambda^3.$$

If we form the covariants $H_{(\kappa,\lambda)}$, and $G_{(\kappa,\lambda)}$, of Ω, where

$$4\Omega \equiv 4\kappa^3 - I\kappa\lambda^2 + J\lambda^3$$

(the reducing cubic rendered homogeneous in κ, λ), we find, as M. Hermite has remarked,

$$I_{(\kappa,\lambda)} = -12H_{(\kappa,\lambda)}, \quad J_{(\kappa,\lambda)} = 4G_{(\kappa,\lambda)}.$$

Again, to calculate the Hessian of $\kappa U - \lambda H_x$, we reduce

$$R_1^2 X^2 + R_2^2 Y^2 + R_3^2 Z^2$$

by the substitutions

$$\rho_1^3 X^2 + \rho_2^3 Y^2 + \rho_3^3 Z^2 \equiv -\tfrac{1}{4} I U,$$

$$\rho_1^4 X^2 + \rho_2^4 Y^2 + \rho_3^4 Z^2 \equiv \tfrac{1}{4}(IH_x + JU),$$

the first of which follows from the equations

$$\rho_1^2 = \rho_2\rho_3 + \tfrac{1}{4}I, \quad \rho_2^2 = \rho_3\rho_1 + \tfrac{1}{4}I, \quad \rho_3^2 = \rho_1\rho_2 + \tfrac{1}{4}I,$$

multiplying by $\rho_1 X^2$, $\rho_2 Y^2$, $\rho_3 Z^2$, respectively, and the second by $\rho_1^2 X^2$, $\rho_2^2 Y^2$, $\rho_3^2 Z^2$; and adding.

In this way we find the following form for the Hessian of $\kappa U - \lambda H_x$:—

$$\frac{1}{4}\left\{H_x\left(4\kappa^2 - \frac{I}{3}\lambda^2\right) - U\left(\frac{2}{3}I\kappa\lambda - J\lambda^2\right)\right\};$$

which may be expressed in the form

$$\frac{1}{3}\left(H_x\frac{d\Omega}{d\kappa} + U\frac{d\Omega}{d\lambda}\right),$$

which is the Jacobian of $\kappa U - \lambda H_x$ and Ω, the variables being κ and λ.

Again, since $\quad I^3 - 27J^2 = 16(\rho_2 - \rho_3)^2(\rho_3 - \rho_1)^2(\rho_1 - \rho_2)^2,$

and $$G_x = \tfrac{1}{2}\sqrt{I^3 - 27J^2}\,.\,XYZ;$$

transforming ρ_1, ρ_2, ρ_3 into R_1, R_2, R_3, we find

$$I^3_{(\kappa,\lambda)} - 27J_{(\kappa,\lambda)} = \Omega^2(I^3 - 27J^2), \quad G_{(\kappa,\lambda)x} = \Omega G_x,$$

We have therefore expressed the invariants and covariants of $\kappa U - \lambda H_x$ in terms of the invariants and covariants of U.

181. **Number of Covariants and Invariants of the Quartic.**—We proceed to prove the following proposition, which determines the number of these functions:—

The quartic has only the two distinct invariants I and J, and two distinct covariants whose leading coefficients are H and G.

This proposition asserts that every invariant is a *rational* and *integral* function of I and J, and every covariant a *rational* and *integral* function of U, H_x, G_x, I, J. The following discussion is founded on principles similar to those already employed in the case of the cubic.

Attending to the observations in Arts. 36, 37, it is plain that if $\phi(a, \beta, \gamma, \delta)$ be any integral function of the differences of the roots expressible by the coefficients in a rational form, we have, in general, considering the equation with the second term removed,

$$a^r\phi(a, \beta, \gamma, \delta) = F(a, H, I, G),$$

where F is a rational and integral function, and r remains to be determined.

And if, in the first place, ϕ be an odd function of the roots; changing their signs, and subtracting the two values of ϕ, we find

$$2a^r\phi(\alpha, \beta, \gamma, \delta) = F(a, H, I, G) - F(a, H, I, -G).$$

This value of ϕ plainly vanishes with G; whence, eliminating the powers of G beyond the first by the identical equation of Art. 37, we have

$$a^r\phi(\alpha, \beta, \gamma, \delta) = GF_1(a, H, I, J).$$

It follows that every odd function ϕ of the differences of the roots is divisible by

$$(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta);$$

and removing this factor on the first side of the equation, and $32\dfrac{G}{a^2}$ on the second side, we have

$$a^{r-3}\phi_1(\alpha, \beta, \gamma, \delta) = F_1(a, H, I, J),$$

where ϕ_1 is an even function of the roots, and F_1 a rational and integral function.

We proceed to prove, in the second place, if $\phi(\alpha, \beta, \gamma, \delta)$ be any even integral function of the differences of the roots, of the order ϖ, expressible by the coefficients in a rational form, that $a^\varpi\phi(\alpha, \beta, \gamma, \delta)$ can be expressed as a rational and *integral* function of a, H, I, J.

To prove this, the following lemma is necessary:—

There exists no function of H, I, J which is divisible by a. For, suppose $F(H, I, J)$ to be divisible by a. Making a vanish, we have $F(H', I', J') = 0$, where $H' = -b^2$, $I' = -4bd + 3c^2$, $J' = 2bcd - eb^2 - c^3$ (the values of H, I, J, when $a = 0$); and as it is impossible to eliminate b, c, d, e, so as to obtain a relation between H', I', J', we conclude that no relation such as $F(H', I', J') = 0$ exists; and therefore there is no function of the form $F(H, I, J)$ which is divisible by a.

We now proceed with the proof of the proposition; and since, as has been already proved in the case of an even function of the roots,

$$a^r\phi(\alpha, \beta, \gamma, \delta) = F(a, H, I, J),$$

we have, dividing by $a^{r-\varpi}$,

$$a^{\varpi}\phi(a, \beta, \gamma, \delta) = F_0(a, H, I, J) + \Sigma\frac{F_p(H, I, J)}{a^p}.$$

Again, since the first side of this equation is expressible as a rational and integral function of the coefficients not divisible by a, the second side must be a similar function of the coefficients; and this, by the lemma just established, is impossible unless such terms as $\Sigma\frac{F_p(H, I, J)}{a^p}$ disappear.

Wherefore

$$a^{\varpi}\phi(a, \beta, \gamma, \delta) = F_0(a, H, I, J);$$

and, finally, we have proved that $a^{\varpi}\phi(a, \beta, \gamma, \delta)$ may be expressed by the forms

$$GF(a, H, I, J), \quad \text{or} \quad F(a, H, I, J),$$

according as ϕ is odd or even.

We are now in a position to prove the original proposition as to the number of invariants and covariants. For, if $F(a, H, I, J)$ be an invariant, a and H must disappear, since if they were present this function could not remain the same when the coefficients are written in direct or reverse order. Similarly, no odd function such as $GF(a, H, I, J)$ can give an invariant. It follows that every invariant is a function of I and J.

Again, the quartic has only two distinct covariants; for we have proved that every function of the differences $a^{\varpi}\phi$ is of one of the forms

$$F(a, H, I, J) \quad \text{or} \quad GF(a, H, I, J).$$

Now, considering these forms as the leading terms of covariants, it has been proved that every covariant is expressible as

$$F(U, H_x, I, J) \quad \text{or} \quad G_xF(U, H_x, I, J);$$

that is, every covariant is expressible in terms of H_x and G_x, along with U, I, and J; and this is the proposition which was required to be proved.

Examples.

1. If U be any cubic, and G_x its cubic covariant, prove that the Hessian of $\lambda U + \mu G_x$ has the same roots as the Hessian of U, λ and μ being constants.

2. Prove that any covariant of a quantic, whose roots are $\alpha_1, \alpha_2, \ldots \alpha_n$, satisfies the equation

$$\Sigma\alpha^2 \frac{d\phi}{d\alpha} - \varpi s_1 \phi = x\frac{d\phi}{dy},$$

where ϖ is the degree of ϕ in the coefficients of the quantic, and $s_1 = \Sigma\alpha$.

3. If a quantic have a square factor, prove that the same square factor enters its Hessian.

4. If a quartic have a square factor, the covariant G_x has that factor as a quintuple factor.

5. Prove that the sextic covariant G_x of the quartic $\phi(x)$ may be written under the form

$$\{\phi(x)\}^2 \Sigma \frac{\phi'(\alpha)}{(x-\alpha)^2}.$$

6. Applying the principles of Art. 180, determine the form of the sextic covariant of the quartic $\lambda U + \mu H_x$.

7. Calculate the values of H, I, G, J for the Hessian of a quartic.

$$Ans. \quad H' = \frac{3a_0 J - HI}{12}, \quad I' = \frac{I^2}{12}, \quad G' = -\frac{JG}{4}, \quad J' = \frac{54J^2 - I^3}{216}.$$

8. Find the two conditions (Ex. 3, p. 125) that the Hessian in the preceding question should be a perfect square, and show that both contain J as a factor.

$$Ans. \; JG = 0, \quad a_0 J(2HI - 3a_0 J) = 0.$$

9. A function ϕ of the differences of the roots of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0$$

arranged in powers of a_n being

$$\phi \equiv A_p + pA_{p-1}a_n + \frac{p \cdot p-1}{1 \cdot 2} A_{p-2}a_n^2 + \ldots + A_0 a_n^p;$$

prove that $DA_j = -na_{n-1}jA_{j-1}$, and hence show that if $\psi(a_0, a_1, a_2, \ldots a_r)$ is a function of the differences so also is $\psi(A_0, A_1, A_2, \ldots A_r)$.

10. Hence show how the final coefficient of the equation of squared differences can be found for any equation when it is known for the equation of next lower order.

11. If the discriminant of a biquadratic be written under the form

$$(A_0,\ A_1,\ A_2,\ A_3)\,(a_4,\ 1)^3,$$

prove that the discriminant of this cubic is

$$27^2 G^2 \Delta_3{}^3,$$

where Δ_3 is the discriminant of $(a_0,\ a_1,\ a_2,\ a_3)\,(x,\ 1)^3$; and knowing A_3, find A_2, A_1, and A_0.

12. Form the equation whose roots are

$$\phi(\alpha_1),\quad \phi(\alpha_2),\quad \phi(\alpha_3),\ \ldots\ \phi(\alpha_n),$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are the roots of $f(x) = 0$, the resultant R of $f(x)$ and $\phi(x)$ being given.

Change the last coefficient b_m of $\phi(x)$ into $b_m - \rho$, and substitute this value for b_m in the equation $R = 0$.

13. Prove that the quartic

$$f(x,\ y) \equiv (a,\ b,\ c,\ d,\ e)\,(x,\ y)^4$$

may be reduced by a linear transformation $x = \lambda X + \mu Y$, $y = \lambda' X + \mu' Y$ to the form

$$f(\lambda,\ \lambda')\,X^4 + f(\mu,\ \mu')\,Y^4 + 6\rho M^2 X^2 Y^2,$$

where

$$4\rho^3 - I\rho + J = 0, \quad M = \lambda\mu' - \lambda'\mu.$$

14. Retaining the notation of the last example, prove that $\frac{\lambda}{\lambda'}$ and $\frac{\mu}{\mu'}$ are the roots of one of the factors u, v, w of the sextic covariant of the quartic.

15. Prove that

$$\frac{d^3 G_x}{dx^3} = 60\,(U_1{}^2 U_4 - U_0 U_3{}^2),$$

the reducing cubic of Art. 65 (cf. Ex. 5, p. 132).

16. Prove that

$$\rho_1{}^p X^2 + \rho_2{}^p Y^2 + \rho_3{}^p Z^2 = \Pi_{p-2} H_x - \Pi_{p-1} U,$$

where Π_{p-1}, Π_{p-2} are sums of homogeneous products (Ex. 6, p. 320).

CHAPTER XVII.

COVARIANTS AND INVARIANTS OF COMBINED FORMS.

182. **Combined Forms.**—In the present chapter we propose to illustrate the theory of the covariants and invariants of systems of two or more quantics (see Art. 159) by the simplest cases, viz.—(1) two quadratics, (2) quadratic and cubic, and (3) two cubics. We give in each case an enumeration of the forms which have been shown to be fundamental by the investigations of Clebsch, Gordan, and Sylvester; showing how these forms may be obtained, but without attempting the reduction of all other forms dependent on them. In estimating the number of covariants and invariants of a combined system, the independent forms which belong to each quantic by itself are counted among the total number belonging to the system. It will be found convenient to use the term *special* to designate those forms which belong to the two quantics *regarded as a system* (and which therefore contain the coefficients of both), as distinguished from those which belong to the quantics taken separately.

Invariants and covariants are both included under the name *concomitant*, which is applied to any function whose relations to the quantics are independent of linear transformation.

183. **Two Quadratics.**—Let the two quadratics be

$$U \equiv a_1x^2 + 2b_1xy + c_1y^2, \quad V \equiv a_2x^2 + 2b_2xy + c_2y^2.$$

This system has one special invariant, and one special covariant. The invariant may be obtained by forming the discriminant of $\lambda U + \mu V$, which is found to be

$$\lambda^2 (a_1c_1 - b_1^2) + \lambda\mu (a_1c_2 + a_2c_1 - 2b_1b_2) + \mu^2 (a_2c_2 - b_2^2),$$

all the coefficients of $\lambda : \mu$ being invariants (Art. 169); whence we have the special invariant

$$a_1c_2 + a_2c_1 - 2b_1b_2 = 2I_{12}. \quad \text{(cf. Ex. 3, Art. 165.)}$$

The vanishing of this function of the coefficients is the condition that the pencil of lines $UV = 0$ should be harmonic, the rays represented by one equation being conjugate to those represented by the other.

The special covariant is the Jacobian of the system, viz.

$$\begin{vmatrix} a_1x + b_1y & b_1x + c_1y \\ a_2x + b_2y & b_2x + c_2y \end{vmatrix} \equiv J(U, V),$$

which may be written in the form

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix},$$

obtained by eliminating dialytically the variables from the quantics U, V, $(xy' - x'y)^2$, the form $xy' - x'y$ being a universal concomitant of all binary quantics (cf. Art. 169).

The square of J is connected with U and V by the following important relation (cf. Ex. 6, p. 389):—

$$-J^2(U, V) = I_{22}U^2 - 2I_{12}UV + I_{22}V^2, \tag{1}$$

which may be derived immediately from the equation

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{vmatrix} x^2 & 2xy & y^2 \\ c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \end{vmatrix} = \begin{vmatrix} 0 & U & V \\ U & 2I_{11} & 2I_{12} \\ V & 2I_{12} & 2I_{22} \end{vmatrix}.$$

It is easy to see that $J(UV)$ gives the double lines of the system $\lambda U + \mu V$, for when $\lambda U + \mu V$ is a perfect square

$$\lambda^2 I_{11} + 2\lambda\mu I_{12} + \mu^2 I_{22} = 0,$$

and eliminating $\lambda : \mu$ by means of the equation $\lambda U + \mu V = 0$, the double lines are determined by the equation

$$I_{22}U^2 - 2I_{12}UV + I_{11}V^2 = 0,$$

or

$$J^2(U, V) = 0.$$

Every concomitant of a system of two quadratics may be expressed in terms of the six forms U, V, $J(U, V)$, I_{11}, I_{12}, I_{22}, all of which are constituents of the formula (1) written above. The resultant of U, V, for example, is

$$4(I_{11}I_{22} - I_{12}^2) \qquad \text{(See Art. 149.)}$$

which is also the discriminant of $J(U, V)$, and the dialytic eliminant of U, V, $J(U, V)$.

184. **Quadratic and Cubic.**—Let the two quantics be

$$U \equiv (a, b, c, d)(x, y)^3, \quad V \equiv (a', b', c')(x, y)^2,$$

the covariants of U being denoted as usual by H_x and G_x. The system has one special cubic covariant, the Jacobian of U and V, or $J(U, V)$; and one special quadratic covariant, viz., $J(H_x, V)$.

In writing down the remaining covariants it will be found convenient to adopt the following notation. We use U with suffix D to denote the result of substituting in U the differential symbols D_y, $-D_x$ for x, y, respectively, where $D_x \equiv \dfrac{d}{dx}$, $D_y \equiv \dfrac{d}{dy}$; hence

$$U_D \equiv (a, b, c, d)(D_y, -D_x)^3, \quad V_D \equiv (a', b', c')(D_y, -D_x)^2,$$

with a corresponding notation in other cases.

There are four linear covariants, which may now be written as follows:—

$$V_D(U), \quad V_D(G_x), \quad U_D(V^2), \quad G_D(V^2).$$

The first of these written at length is

$$(ac' - 2bb' + ca')\,x + (bc' - 2cb' + da')\,y.$$

There are three special invariants. The first is the intermediate invariant of the system of two quadratics H_x and V, viz.,

$$(ac - b^2)\,c' - (ad - bc)\,b' + (bd - c^2)\,a' = I_{21},$$

where the notation I_{pq} is used to signify that the invariant is of the p^{th} degree in the coefficients of U and the q^{th} in the coefficients of V. The second invariant is the resultant R of U and V. It is of the second degree in the coefficients of U, and third in the coefficients of V, and may be expressed in many ways by the methods of elimination of Chap. XIV. The general form of any invariant I_{23} of this type is

$$I_{23} = lR + m\,(a'c' - b'^2)\,I_{21},$$

l and m being numbers.

The third invariant is of the type I_{43}, and may be obtained by operating with V three times in succession on the product of U and G_x; it can be written in the form

$$V_D{}^3\,(UG_x).$$

There are, therefore, nine special forms belonging to this system; and if to these be added U and V, and the independent covariants and invariants of each, we obtain the complete list of fifteen forms, viz., three cubic, three quadratic, and four linear covariants, and five invariants.

185. **Two Cubics.**—Let the cubics be

$$U = (a, b, c, d)(x, y)^3, \quad V = (a', b', c', d')(x, y)^3,$$

the covariants of U being represented as before by H_x and G_x, and those of V by H_x' and G_x'.

Of this system there is one quartic covariant, the Jacobian of U and V, viz.,

$$J(U, V) = (ab')x^4 + 2\,(ac')\,x^3y + \{(ad') + 3\,(bc')\}\,x^2y^2$$
$$+ 2\,(bd')\,xy^3 + (cd')\,y^4;$$

and two special cubic covariants, viz. :—

$$J(U, H'_x), \text{ and } J(V, H_x).$$

There are four special quadratic covariants. If we form the Hessian of $\lambda U + \mu V$, *i.e.* substitute $\lambda a + \mu a'$, $\lambda b + \mu b'$, &c., for a, b, &c. in H_x, we find

$$\lambda^2 H_x + \lambda\mu K_x + \mu^2 H'_x.$$

The intermediate Hessian K_x here obtained is the first special quadratic covariant; and the remaining three are obtained by taking the Jacobians in pairs of H_x, K_x, and H_x'.

There are six linear covariants which may be written as follows:—

$$H_D(V),\ H_D(G'_x),\ H'_D(U),\ H'_D(G_x),\ U_D(H'^2_x), V_D(H^2_x).$$

It is easily seen that $H_D(U)$ and $H_D(G_x)$ vanish identically, for U and G_x may be brought by linear transformation to the forms $ax^3 + dy^3$, and $ad(ax^3 - dy^3)$, respectively, and H_x to the form $adxy$ (cf. Art. 173).

There are in all seven invariants, five of which may be obtained by forming the discriminant of $\lambda U + \mu V$, the coefficients of the different powers of $\lambda : \mu$ being invariants. If the discriminant is

$$\lambda^4\Delta + 4\lambda^3\mu\Theta + 6\lambda^2\mu^2\Phi + 4\lambda\mu^3\Theta' + \mu^4\Delta',$$

we obtain in this way three special invariants Θ, Φ, Θ', the extreme coefficients being the discriminants of U and V. The two remaining invariants are of odd orders in the coefficients of each cubic. They are denoted by P and Q, and may be defined as follows:—

$$P \equiv \tfrac{1}{6}U_D(V) = (ad') - 3\,(bc'), \qquad (1)$$

$$27Q \equiv P^3 - R, \qquad (2)$$

where R is the resultant of U and V as obtained by Bezout's method (Art. 154), viz.

$$R \equiv (ad')^3 - 18\,(ab')(cd')(ad') + 9\,(bd')(ca')(ad')$$
$$+ 27\,(ca')^2(cd') + 27\,(ab')(bd')^2 - 81\,(ab')(bc')(cd').$$

Substituting this value of R in (2), we find

$$-Q \equiv (bc')^3 + (ca')^2(cd') + (ab')(bd')^2 - (bc')^2(ad')$$
$$- 3\,(ab')(bc')(cd') - (ad')(ab')(cd').$$

Any invariant comprised in the formula $lP^3 + mR$, where l and m are numbers, being of the type I_{33}, might have been selected instead of Q as the fundamental invariant of this type; reasons will appear subsequently for the selection which has been made (see Ex. 4, p. 416).

If to the special forms enumerated be added those which belong to each cubic, we have in all twenty-six fundamental forms, viz. one quartic, six cubic, six quadratic, and six linear, covariants; and seven invariants.

Several of the covariants and invariants enumerated in the preceding Articles will be found expressed in terms of the roots of the two equations of the combined system among the examples which follow on the next page.

186. **Combinants.**—Combined forms give rise to a series of invariants and covariants whose coefficients are expressible by determinants of the form $(a_r b_s)$, such as occur in the resultant obtained by Bezout's method (Art. 154). These concomitants are unaltered (save by a factor of the form $(\lambda\mu' - \lambda'\mu)^r$) when the quantics U, V are changed into $\lambda U + \mu V$, $\lambda' U + \mu' V$. Such invariants have been called *combinants*, and the corresponding covariants may be termed in like manner *combining covariants*. Of the former we have examples in P and Q of the preceding Article; and Jacobians in general are examples of the latter class of concomitants.

It may be noticed that the I and J invariants of the biquadratic in $\lambda : \mu$ of the preceding Article, viz. the discriminant of $\lambda U + \mu V$, are combinants of the system of two cubics; for in fact a linear transformation of λ and μ is equivalent to a transformation of U and V of the kind considered in the present Article, and therefore any function of the invariants Δ, Θ, Φ, &c., unaltered by such transformation must be a combinant. It can be verified that these invariants may be expressed in terms of P and Q as follows (see Salmon's *Higher Algebra*, Art. 218):—

$$I = 3P(P^3 - 24Q), \quad J = -P^6 + 36P^3Q - 216Q^2.$$

EXAMPLES.

1. If α, β, γ, and α', β' are the roots of the equations

$$U \equiv ax^3 + 3bx^2 + 3cx + d = 0, \quad V \equiv a'x^2 + 2b'x + c' = 0\,;$$

express in terms of the coefficients the function

$$(\beta - \gamma)^2 (\alpha - \alpha') (\alpha - \beta') + (\gamma - \alpha)^2 (\beta - \alpha') (\beta - \beta') + (\alpha - \beta)^2 (\gamma - \alpha') (\gamma - \beta').$$

Denoting this function by ϕ, we easily find

$$- a^2 a' \phi = 9 \{ a' (bd - c^2) - b' (ad - bc) + c' (ac - b^2) \}.$$

The given function of the roots is an invariant of the system, for it involves all the roots of the cubic in the second degree, and all the roots of the quadratic in the first degree. If, in fact, we make the substitutions of Art. 159, and multiply by $U^2 V$ to make the function integral, the result will not contain x, and is therefore an invariant (cf. Art. 184).

The geometrical interpretation of the equation $\phi = 0$ is that the quadratic V should form with the Hessian of U a harmonic system.

2. Using the same notation as in the preceding question, find the condition that one pair of roots of $U = 0$ should form a harmonic range with the roots of $V = 0$.

Ans. $R + 9 (a'c' - b'^2) I_{21} = 0$.

3. If α, β, γ, and α', β', γ' be the roots of the cubics

$$U \equiv ax^3 + 3bx^2 + 3cx + d = 0, \quad V \equiv a'x^3 + 3b'x^2 + 3c'x + d' = 0,$$

express the following function (when multiplied by aa') in terms of the coefficients, and prove that it is an invariant of the system :—

$$(\alpha - \alpha') (\beta - \beta') (\gamma - \gamma') + (\alpha - \beta') (\beta - \gamma') (\gamma - \alpha') + (\alpha - \gamma') (\beta - \alpha') (\gamma - \beta')\,;$$

or, differently arranged,

$$(\alpha - \alpha') (\beta - \gamma') (\gamma - \beta') + (\alpha - \beta') (\beta - \alpha') (\gamma - \gamma') + (\alpha - \gamma') (\beta - \beta') (\gamma - \alpha').$$

Ans. $3P$, where $P \equiv (ad' - a'd) - 3 (bc' - b'c)$. (cf. Art. 185.)

4. Retaining the notation of the preceding example, prove that if κ can be determined so as to make $U + \kappa V$ a perfect cube, the following relation exists among the roots of the two cubics :—

$$(\beta - \gamma) \sqrt[3]{\phi(\alpha)} + (\gamma - \alpha) \sqrt[3]{\phi(\beta)} + (\alpha - \beta) \sqrt[3]{\phi(\gamma)} = 0,$$

where $\phi(x) \equiv V$, and α, β, γ are the roots of $U = 0$; and prove that in this case the invariant Q (see Art. 185) vanishes.

The relation among the roots is obtained immediately by substituting α, β, γ for x in the identity $U + \kappa V \equiv (lx + m)^3$, and eliminating κ, l, m from the resulting equations.

Rationalising, we have

$$\left\{\frac{(\beta-\gamma)^3\,\phi(\alpha)+(\gamma-\alpha)^3\,\phi(\beta)+(\alpha-\beta)^3\,\phi(\gamma)}{(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta)}\right\}^3-27\,\phi(\alpha)\,\phi(\beta)\,\phi(\gamma)=0.$$

Substituting for $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$; introducing the relations obtained by comparing the different powers of λ in the following identity:—

$$\Sigma(\alpha+\lambda)^3\,(\beta-\gamma)^3\equiv 3\,(\alpha+\lambda)\,(\beta+\lambda)\,(\gamma+\lambda)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta);$$

and expressing the result in terms of the coefficients, we find

$$\{3P\}^3-27\,R=0, \quad \text{or } Q=0 \text{ (see Art. 185).}$$

We now give several different forms under which the invariant Q presents itself. Since $U+\kappa V$ is a perfect cube, we have (Art. 43)—

$$\frac{a+\kappa a'}{b+\kappa b'}=\frac{b+\kappa b'}{c+\kappa c'}=\frac{c+\kappa c'}{d+\kappa d'}. \qquad (1)$$

Equating these fractions separately to $-\kappa'$, we find

$$\begin{aligned} a+\kappa a'+\kappa' b+\kappa\kappa' b' &= 0,\\ b+\kappa b'+\kappa' c+\kappa\kappa' c' &= 0,\\ c+\kappa c'+\kappa' d+\kappa\kappa' d' &= 0; \end{aligned} \qquad (2)$$

and solving for κ, κ', $\kappa\kappa'$, we may eliminate them, and find the condition in the form

$$Q\equiv\begin{vmatrix} a & b & c\\ a' & b' & c'\\ b' & c' & d'\end{vmatrix}\begin{vmatrix} b' & c' & d'\\ a & b & c\\ b & c & d\end{vmatrix}-\begin{vmatrix} a' & b' & c'\\ a & b & c\\ b & c & d\end{vmatrix}\begin{vmatrix} b & c & d\\ a' & b' & c'\\ b' & c' & d'\end{vmatrix}=0.$$

Again, eliminating κ and κ^2 from the equations (1) without introducing κ', we obtain another form for Q, viz.

$$Q\equiv\begin{vmatrix} ac-b^2 & ac'+a'c-2bb' & a'c'-b'^2\\ ad-bc & ad'+a'd-bc'-b'c & a'd'-b'c'\\ bd-c^2 & bd'+b'd-2cc' & b'd'-c'^2\end{vmatrix}=0.$$

This form of Q can be readily obtained also by expressing the condition that the Hessian of $\lambda U+\mu V$ (see Art. 185) should vanish identically—a condition which is fulfilled when $\lambda U+\mu V$ is a perfect cube.

Finally, writing the equations (2) in the form

$$\frac{a+\kappa' b}{a'+\kappa' b'}=\frac{b+\kappa' c}{b'+\kappa' c'}=\frac{c+\kappa' d}{c'+\kappa' d'},$$

and eliminating κ' and κ'^2, we have a third form for Q, viz.

$$Q\equiv\begin{vmatrix} (ab') & (ac') & (bc')\\ (ac') & (ad')+(bc') & (bd')\\ (bc') & (bd') & (cd')\end{vmatrix}=0.$$

The constituents in this form are the same minor determinants that occur in Bezout's form of the resultant, and it may be easily verified that this value of Q agrees with the expanded form written in Art. 185.

5. Find the condition that the roots of two cubics should determine a system in involution.

The condition in terms of the roots is expressed by equating to zero the product of six determinants of the type

$$\begin{vmatrix} 1 & \alpha+\alpha' & \alpha\alpha' \\ 1 & \beta+\beta' & \beta\beta' \\ 1 & \gamma+\gamma' & \gamma\gamma' \end{vmatrix}.$$

6. Express the condition of the preceding example in terms of the coefficients of the cubics.

The roots of one cubic being conjugates to the roots of the other, the two are reducible to the following forms :—

$$U \equiv ax^3 + 3bx^2 + 3cx + d,$$

$$V \equiv dx^3 + 3\kappa cx^2 + 3\kappa^2 bx + \kappa^3 a;$$

and writing the discriminant of $\rho U + V$ in general in the form (see Art. 185)—

$$\rho^4\Delta + 4\rho^3\Theta + 6\rho^2\Phi + 4\rho\Theta' + \Delta',$$

we find in this case

$$\Theta' = \kappa^3\Theta, \quad \Delta' = \kappa^6\Delta;$$

whence the required condition

$$\Delta\Theta'^2 - \Delta'\Theta^2 = 0.$$

7. Express in terms of the coefficients of the cubics of Ex. 3 the following covariant of the system :—

$$aa'\Sigma\{3(\beta-\beta')(\gamma-\gamma')+3(\beta-\gamma')(\gamma-\beta')+(\beta-\gamma)(\beta'-\gamma')\}(x-\alpha)(x-\alpha').$$

Ans. $18\{(ac'+a'c-2bb')x^2+(ad'+a'd-bc'-b'c)x+(bd'+b'd-2cc')\}$

8. To reduce the two cubics

$$U \equiv (a, b, c, d)(x, y)^3, \quad V \equiv (a', b', c', d')(x, y)^3$$

to the forms

$$U = \frac{1}{4}\frac{dF}{dX}, \quad V = \frac{1}{4}\frac{dF}{dY},$$

by means of a linear transformation

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

the coefficients in which are to be determined in terms of the coefficients of the given cubics.

Let $\quad F = (A, B, C, D, E)(X, Y)^4;$

then $\quad U \equiv (a, b, c, d)(x, y)^3 = (A, B, C, D)(X, Y)^3,$

$$V \equiv (a', b', c', d')(x, y)^3 = (B, C, D, E)(X, Y)^3.$$

Now, substituting the differential symbols $D_y, -D_x$ for x, and $\frac{1}{M}D_Y, -\frac{1}{M}D$ for X and Y in the Hessian of both forms of U, we find the operational equation

$$\begin{vmatrix} D^2_{x^2} & D^2_{xy} & D^2_{y^2} \\ a & b & \\ b & c & d \end{vmatrix} = \frac{1}{M^4} \begin{vmatrix} D^2_{X^2} & D^2_{XY} & D^2_{Y^2} \\ A & B & C \\ B & C & D \end{vmatrix};$$

whence, operating on both forms of V, we have

$$\psi(x, y) \equiv \begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix} x + \begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} y = \frac{JY}{M^4}.$$

Similarly,

$$\phi(x, y) \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} x + \begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} y = JX,$$

where ϕ and ψ are covariants of U and V, and J is the ternary invariant of F.

Again, since

$$\phi(D_y, -D_x) = \frac{J}{M^5} D_Y, \quad \text{and} \quad -\psi(D_y, -D_x) = \frac{J}{M^5} D_X,$$

performing the operation

$$\phi(D_y, -D_x)\,\psi(x, y), \quad \text{or} \quad \psi(D_y, -D_x)\,\phi(x, y),$$

on equivalent forms, we have

$$Q \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} \begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} - \begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix} \begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} = \frac{J^2}{M^9}.$$

We are now in a position to determine the coefficients of F in terms of the coefficients of U and V.

For we have from former equations

$$Qx = \begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} \phi - \begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} \psi,$$

$$Qy = -\begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix} \phi + \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} \psi;$$

whence, substituting these values of x and y in U and V, we find

$$Q^3 U = (A_0, B_0, C_0, D_0)(\phi, \psi)^3,$$

$$Q^3 V = (B_0, C_0, D_0, E_0)(\phi, \psi)^3,$$

and, therefore,

$$Q^3 U = \frac{1}{4}\frac{dF_0}{d\phi}, \quad Q^3 V = \frac{1}{4}\frac{dF_0}{d\psi}, \quad \text{where} \quad F_0 = (A_0, B_0, C_0, D_0, E_0)(\phi, \psi)^4;$$

also

$$\frac{A}{A_0} = \frac{B}{B_0} = \frac{C}{C_0} = \frac{D}{D_0} = \frac{E}{E_0} = \frac{M^{15}}{J^5}.$$

9. Determine the invariants of F_0 in the preceding example, and hence infer the form of the resultant of two cubics.

We have, from the equations of Ex. 8,

$$J^{10} = M^{45} J_0, \quad \text{and} \quad J^6 I = M^{30} I_0;$$

and, substituting differential symbols for x, y and X, Y in both forms of V, and operating on U, we find

$$P \equiv ad' - a'd - 3(bc' - b'c) = \frac{I}{M^3},$$

which equation, along with the equation $Q = \frac{J^2}{M^9}$, enables us by previous results to express I_0 and J_0 in terms of P and Q in the following way:—

$$I_0 = PQ^3, \quad \text{and} \quad J_0 = Q^5.$$

From these results we derive the relations

$$\frac{I_0{}^3}{J_0{}^2} = \frac{I^3}{J^2} = \frac{P^3}{Q},$$

from which it follows that when $I^3 = 27J^2$, we have $P^3 = 27Q$; but the first relation holds when F has a square factor, which necessitates U and V having a common factor; whence we infer that $P^3 - 27Q$, being of the proper degree and weight, is the resultant of the cubics U and V (cf. Art. 185).

10. If $\alpha, \beta, \gamma, \delta; \alpha', \beta', \gamma', \delta'$ be the roots of the biquadratics

$$(a, b, c, d, e)(x, 1)^4 = 0, \quad (a', b', c', d', e'(x, 1)^4 = 0,$$

prove

$$aa' \Sigma (\alpha - \alpha')(\beta - \beta')(\gamma - \gamma')(\delta - \delta') = 24\{ae' + a'e - 4(bd' + b'd) + 6cc'\},$$

and show that this function is an invariant of the system.

11. Prove that the following function of the roots of a biquadratic and quadratic gives an invariant of the system, and determine its geometrical interpretation:—

$$\begin{vmatrix} 1 & \beta+\gamma & \beta\gamma \\ 1 & \alpha+\delta & \alpha\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \times \begin{vmatrix} 1 & \gamma+\alpha & \gamma\alpha \\ 1 & \beta+\delta & \beta\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \times \begin{vmatrix} 1 & \alpha+\beta & \alpha\beta \\ 1 & \gamma+\delta & \gamma\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \equiv \phi.$$

The geometrical interpretation of the equation $\phi = 0$ is, that the two conjugate foci of some one of the three involutions determined by the biquadratic form along with the quadratic an harmonic system.

12. Prove that the following functions of the roots of a biquadratic and quadratic give invariants of the system, and determine their values in terms of the coefficients:—

$$a_0 b_0^2 \Sigma (\alpha' - \alpha)(\alpha' - \beta)(\beta' - \gamma)(\beta' - \delta),$$

$$a_0^2 b_0^2 \Sigma (\alpha - \beta)^2 (\gamma - \alpha')(\delta - \beta')(\gamma - \beta')(\delta - \alpha').$$

13. If $f(x)$ and $\phi(x)$ be two quartics with unequal roots, the roots of $f(x)$ being $\alpha, \beta, \gamma, \delta$, prove that the condition that a quartic of the system $\lambda f(x) + \mu\phi(x)$ can have two square factors may be expressed as follows:—

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \sqrt{\phi(\alpha)} \\ 1 & \beta & \beta^2 & \sqrt{\phi(\beta)} \\ 1 & \gamma & \gamma^2 & \sqrt{\phi(\gamma)} \\ 1 & \delta & \delta^2 & \sqrt{\phi(\delta)} \end{vmatrix} = 0.$$

14. Determine the condition in terms of the coefficients that the quartic $\lambda f(x) + \mu\phi(x)$ may have two square factors.

In this case the Hessian of $\lambda f(x) + \mu\phi(x) = \kappa \{\lambda f(x) + \mu\phi(x)\}$, from which identity we have five equations to eliminate $\lambda^2, \lambda\mu, \mu^2, \kappa\lambda, \kappa\mu$; thus obtaining an invariant I_{44}, of the 4^{th} degree in the coefficients of each equation.

15. Prove that the resultant of two quartics becomes a perfect square when the invariant I_{44} vanishes.

Rendering rational the determinant in Ex. 13, and dividing by the product of the squares of the differences of the roots, we find, when the coefficients are introduced,

$$I_{44} \equiv I^2_{22} - 64R; \text{ whence, \&c. \&c.}$$

16. The discriminant of $\lambda U + \mu V$, where U and V are cubics $(a, b, c, d,)(xy)^3$, $(a', b', c', d',)(xy)^3$, being written as in Art. 185, resolve into its factors the covariant

$$(\Delta, \Theta, \Phi, \Theta', \Delta')(V, -U)^4.$$

The leading coefficient of this covariant is easily obtained by forming the discriminant of $aV - a'U$ directly; it is

$$(ab')^2 \{4(ab')(ad') - 3(ac')^2\},$$

which may be written in the form $2A^2 \{PA + 6(AC - B^2)\}$, where A, B, C are the first three coefficients of the Jacobian; and, consequently, the given covariant is expressed as follows:—

$$2J^2(U, V)\{PJ(U, V) + 6 \text{ Hessian of } J(U, V)\}.$$

17. Express the invariants of the Jacobian of two cubics in terms of P and Q.

Ans. $12I' = P^2, \quad 216J' = 54Q - P^3.$

CHAPTER XVIII.

TRANSFORMATIONS.

SECTION I.—TSCHIRNHAUSEN'S TRANSFORMATION.

187. Under the general heading of this chapter we purpose collecting several propositions which could not have been conveniently given elsewhere, and which are of importance in connexion with the subjects discussed in the foregoing pages. We commence with a general theorem relating to rational transformations.

Theorem.—*The most general rational algebraic transformation of a root of an equation of the n^{th} degree can be reduced to an integral transformation of the degree $n-1$ at most.*

For every rational function of a root a_r of the equation $f(x) = 0$ is of the form

$$\frac{\chi(a_r)}{\psi(a_r)},$$

where χ and ψ are integral functions; also,

$$\frac{\chi(a_r)}{\psi(a_r)} = \chi(a_r)\frac{\psi(a_1)\ldots\psi(a_{r-1})\psi(a_{r+1})\ldots\psi(a_n)}{\psi(a_1)\psi(a_2)\ldots\ldots\psi(a_{n-1})\psi(a_n)},$$

and the denominator $\psi(a_1)\psi(a_2)\ldots\psi(a_n)$, being a symmetric function of the roots of $f(x) = 0$, can be expressed as a rational function of the coefficients. Whence $\dfrac{\chi(a_r)}{\psi(a_r)}$ is reduced to an integral form.

Moreover, the numerator of the former fraction is a symmetric function of the roots of the equation $\dfrac{f(x)}{x - a_r} = 0$, and may consequently be expressed as a rational function of the coefficients of that equation; that is, in terms of a_r and the coefficients of $f(x)$.

Now, denoting by $F(a_r)$ this integral form of $\frac{\chi(a_r)}{\psi(a_r)}$, we have by division

$$F(a_r) = Qf(a_r) + \phi(a_r) = \phi(a_r),$$

where $\phi(a_r)$ does not exceed the degree $n-1$; which proves the proposition.

In the particular cases of the quadratic and cubic it follows that the most general rational function of a root can be reduced to a linear function, and a quadratic function of that root, respectively. In the case of the cubic this quadratic function may be reduced to another form which is often useful, as follows:—Denoting the quadratic function by $\psi(\theta)$, and dividing the cubic $f(\theta)$ by $\psi(\theta)$, we have

$$f(\theta) = (q_0 + q_1\theta)\,\psi(\theta) + r_0 + r_1\theta = 0;$$

proving that

$$\psi(\theta) = -\frac{r_0 + r_1\theta}{q_0 + q_1\theta};$$

whence it appears that *the most general transformation of a roo of a cubic may be reduced to a homographic transformation.*

In connexion with the proposition here established it is easy to justify the remarks made in Arts. 59, 66, relative to the solutions of the cubic and the biquadratic equations. With this object in view, let ϕ and ψ be two rational functions of n quantities $a_1, a_2, \ldots a_n$ (which may be considered as the roots of an equation), each having only p values when the roots are interchanged in every way. Denoting these values of both functions obtained by the same substitution by

$$\phi_1, \phi_2, \phi_3, \ldots \phi_p,$$

$$\psi_1, \psi_2, \psi_3, \ldots \psi_p,$$

we have, for every integer j,

$$\phi_1\psi_1^j + \phi_2\psi_2^j + \phi_3\psi_3^j + \ldots + \phi_p\psi_p^j = T_j;$$

a symmetric function of the roots, since it is the sum of all the possible values which $\phi\psi^j$ can take.

In this way we obtain the system of equations

$$\begin{array}{l} \phi_1 + \phi_2 + \phi_3 + \ldots + \phi_p = T_0, \\ \phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \ldots + \phi_p\psi_p = T_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \phi_1\psi_1^{p-1} + \phi_2\psi_2^{p-1} + \phi_3\psi_3^{p-1} + \ldots + \phi_p\psi_p^{p-1} = T_{p-1}, \end{array}$$

where $T_0, T_1, \ldots T_{p-1}$ are all symmetric functions of $a_1, a_2, a_3, \ldots a_n$.

Solving these equations, we find at once ϕ_1 expressed as a symmetric function of $\psi_2, \psi_3, \ldots \psi_{p-1}$; and therefore by the present proposition reducible to a rational and integral function of ψ_1 of the degree $p-1$, since ψ has only p values considered as a function of $a_1, a_2, \ldots a_n$. Now considering the special cases referred to—(1), when $p = 2$, and $n = 3$, it is proved that a linear relation connects ϕ and ψ in terms of symmetric functions of a_1, a_2, a_3; and (2), when $p = 3$, and $n = 4$, ϕ and ψ are in a similar manner shown to be connected by a rational homographic relation.

188. **Formation of the Transformed Equation.**—The transformation explained in the preceding Article was first employed by Tschirnhausen for the reduction of the cubic and biquadratic. We proceed to explain the method of forming in general the equation whose roots are

$$\phi(a_1), \phi(a_2), \phi(a_3), \ldots \phi(a_n),$$

where $\phi(x)$ is a rational and integral function of x of the degree $n-1$.

Let $$\phi(x) \equiv a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1}.$$

Raising $\phi(x)$ to the different powers $2, 3, \ldots n$ in succession, and reducing the exponents of x in each case below n (by dividing by $f(x)$ and retaining the remainder), we have

$$\begin{array}{l} \phi^2 = b_0 + b_1x + b_2x^2 + \ldots + b_{n-1}x^{n-1}, \\ \phi^3 = c_0 + c_1x + c_2x^2 + \ldots + c_{n-1}x^{n-1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \phi^n = l_0 + l_1x + l_2x^2 + \ldots + l_{n-1}x^{n-1}. \end{array}$$

Substituting for x in these equations each of the roots of the equation $f(x) = 0$, and adding, we find, if S_1, S_2, S_3, &c., denote the sums of the powers of the roots of the required equation

$$S_1 = na_0 + a_1 s_1 + a_2 s_2 + \ldots + a_{n-1} s_{n-1},$$

$$S_2 = nb_0 + b_1 s_1 + b_2 s_2 + \ldots + b_{n-1} s_{n-1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$S_n = nl_0 + l_1 s_1 + l_2 s_2 + \ldots + l_{n-1} s_{n-1}.$$

Now, expressing $s_1, s_2, \ldots s_{n-1}$ in terms of the coefficients of $f(x)$, we have $S_1, S_2, \ldots S_n$ determined in terms of the coefficients of $\phi(x)$ and $f(x)$; we are also enabled by Art. 143 to express the coefficients of the equation whose roots are $\phi(a_1)$, $\phi(a_2), \ldots \phi(a_n)$ in terms of $S_1, S_2, \ldots S_n$, and therefore finally in terms of the coefficients of $\phi(x)$ and $f(x)$; thus theoretically the transformation is completed.

189. **Second Method of forming the Transformed Equation.**—There is another way of finding the final equation in ϕ by elimination, which we now give. Since

$$a_0 - \phi + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} = 0,$$

if this equation be multiplied by $x, x^2, \ldots x^{n-1}$, and the exponents of x reduced below n by means of the equation $f(x) = 0$, we have in all n equations to eliminate dialytically the $n - 1$ quantities $x, x^2, \ldots x^{n-1}$. We thus obtain the transformed equation in the form of a determinant of the n^{th} order, ϕ entering into the diagonal constituents only. For example, if $f(x) = x^n - 1$, we obtain the transformed equation in the following form:—

$$\begin{vmatrix} a_0 - \phi & a_1 & a_2 \;.\;.\; a_{n-1} \\ a_{n-1} & a_0 - \phi & a_1 \;.\;.\; a_{n-2} \\ . & . & . \;\;.\;\;.\;\;.\;\;. \\ . & . & . \;\;.\;\;.\;\;.\;\;. \\ a_1 & a_2 & a_3 \;.\;.\; a_0 - \phi \end{vmatrix} = 0.$$

Although these methods of performing Tschirnhausen's

transformation appear simple, yet if they be applied to particular cases the result usually appears in a complicated form. Professor Cayley, by choosing a form of the transformation suggested by M. Hermite, was enabled to take advantage of the theory of covariants, and thus to complete the transformation for the cubic, quartic, and quintic. We shall content ourselves with showing in an elementary way how Professor Cayley's results for the cubic and quartic may be obtained.

190. **Tschirnhausen's Transformation applied to the Cubic.**—Let the cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

be written under the form

$$z^3 + 3Hz + G = 0;$$

and let it be transformed by the substitution

$$y = \lambda + \kappa z + z^2.$$

If z_1, z_2, z_3 be the roots of the cubic, and y_1, y_2, y_3 the corresponding values of y, we have

$$\begin{aligned} y_2 - y_3 &= (z_2 - z_3)(\kappa - z_1), \\ y_3 - y_1 &= (z_3 - z_1)(\kappa - z_2), \qquad (1) \\ y_1 - y_2 &= (z_1 - z_2)(\kappa - z_3), \end{aligned}$$

and consequently,

$$\begin{aligned} 2y_1 - y_2 - y_3 &= (2z_1 - z_2 - z_3)\kappa + (2z_2z_3 - z_3z_1 - z_1z_2), \\ 2y_2 - y_3 - y_1 &= (2z_2 - z_3 - z_1)\kappa + (2z_3z_1 - z_1z_2 - z_2z_3), \qquad (2) \\ 2y_3 - y_1 - y_2 &= (2z_3 - z_1 - z_2)\kappa + (2z_1z_2 - z_2z_3 - z_3z_1). \end{aligned}$$

Wherefore, if the equation in y with the second term removed be

$$Y^3 + 3H'Y + G' = 0,$$

we have from equations (1) and (2)

$$H' = H_\kappa, \quad G' = G_\kappa,$$

where H and G_κ are the Hessian and cubic covariant of

$$\kappa^3 + 3H\kappa + G;$$

and the transformation is therefore completed, since $y_1 + y_2 + y_3$ can be easily determined.

191. **Tschirnhausen's Transformation applied to the Quartic.**—In this case we do not attempt to form directly the transformed quartic, but prove the following theorem, which shows how this transformation may be resolved into two others.

Theorem.—*Tschirnhausen's transformation changes a quartic U into one having the same invariants as $lU + mH_x$, and therefore in general reducible to the latter form by linear transformation.*

To prove this, let the quartic

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$$

be transformed by the substitution

$$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

If x_1, x_2, x_3, x_4 be the roots of the quartic, and y_1, y_2, y_3, y_4 the corresponding values of y, we have

$$\frac{y_2 - y_3}{x_2 - x_3} = a_1 + a_2(x_2 + x_3) + a_3(x_2^2 + x_2x_3 + x_3^2),$$

$$\frac{y_1 - y_4}{x_1 - x_4} = a_1 + a_2(x_1 + x_4) + a_3(x_1^2 + x_1x_4 + x_4^2).$$

From these equations we proceed to show that

$$\frac{(y_2 - y_3)(y_1 - y_4)}{(x_2 - x_3)(x_1 - x_4)} = P_0 + Q_0(x_2x_3 + x_1x_4),$$

where P_0 and Q_0 involve the roots of the quartic symmetrically.

In the first place, we find

$$(x_2^2 + x_2x_3 + x_3^2)(x_1^2 + x_1x_4 + x_4^2) = p_2^2 - p_1p_3 + p_4 - p_2\lambda,$$

where λ has its usual value, viz. $x_2x_3 + x_1x_4$; and secondly, since

$$x_2^2 + x_2x_3 + x_3^2 = (x_2 + x_3)^2 - x_2x_3, \text{ \&c.},$$

we find again,

$$(x_2 + x_3)(x_1^2 + x_1x_4 + x_4^2) + (x_1 + x_4)(x_2^2 + x_2x_3 + x_3^2) = p_3 - p_1p_2 + p_1\lambda.$$

Finally, since the other terms in the product are obviously of the same form as $P_0 + Q_0\lambda$, we have proved that

$$\frac{(y_2 - y_3)(y_1 - y_4)}{(x_2 - x_3)(x_1 - x_4)} = P_0 + Q_0 (x_2x_3 + x_1x_4);$$

whence

$$(y_2 - y_3)(y_1 - y_4) = (\nu - \mu)(P_0 + Q_0\lambda).$$

Now, introducing ρ_1, ρ_2, ρ_3, in place of λ, μ, ν, this and the similar equations preserve their forms; whence, altering P_0 and Q_0 into similar quantities, we obtain the equations

$$(y_2 - y_3)(y_1 - y_4) = 4(\rho_3 - \rho_2)(P - Q\rho_1),$$

$$(y_3 - y_1)(y_2 - y_4) = 4(\rho_1 - \rho_3)(P - Q\rho_2),$$

$$(y_1 - y_2)(y_3 - y_4) = 4(\rho_2 - \rho_1)(P - Q\rho_3),$$

which lead at once to the invariants of the transformed quartic; and comparing their values with the invariants of $\kappa U - \lambda H_x$ given in Art. 180, the theorem follows at once.

192. **Reduction of the Cubic to a Binomial form by Tschirnhausen's Transformation.**—Let the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

be reduced to the form $y^3 - V$ by the transformation

$$y = q + px + x^2.$$

If x_1, x_2, x_3 be the roots of the given cubic, and y_1 a root of the transformed cubic, we have the following equations to determine p and q:—

$$x_1^2 + px_1 + q = y_1,$$

$$x_2^2 + px_2 + q = \omega y_1,$$

$$x_3^2 + px_3 + q = \omega^2 y_1;$$

from which we find

$$p = -\frac{x_1^2 + \omega x_2^2 + \omega^2 x_3^2}{x_1 + \omega x_2 + \omega^2 x_3},\ q = -\tfrac{1}{3}(s_2 + ps_1).$$

Adding $x_1 + x_2 + x_3$ to this value of p, we have

$$p + x_1 + x_2 + x_3 = -\frac{x_2x_3 + \omega x_3x_1 + \omega^2 x_1x_2}{x_1 + \omega x_2 + \omega^2 x_3};$$

it follows (see Ex. 25, p. 57) that there are only two ways of completing this transformation, as the values of p, q ultimately depend on the solution of the Hessian of the cubic.

193. **Reduction of the Quartic to a Trinomial Form by Tschirnhausen's Transformation.**—Let the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

be reduced to the form $y^4 + Py^2 + Q$, in which the second and fourth terms are absent, by the transformation

$$y = q + px + x^2.$$

If x_1, x_2, x_3, x_4 be the roots of the quartic; also y_1, y_2 two *distinct* roots of the transformed quartic, we have the following equations to determine p and q:—

$$x_1^2 + px_1 + q = y_1, \qquad x_3^2 + px_3 + q = y_2,$$

$$x_2^2 + px_2 + q = -y_1, \qquad x_4^2 + px_4 + q = -y_2;$$

from which we find

$$p = -\frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{x_1 + x_2 - x_3 - x_4}, \quad q = -\tfrac{1}{4}(s_2 + ps_1).$$

And, adding $x_1 + x_2 + x_3 + x_4$ to this value of p, we have

$$p + x_1 + x_2 + x_3 + x_4 = \frac{2(x_1x_2 - x_3x_4)}{x_1 + x_2 - x_3 - x_4};$$

hence, by Ex. 5, p. 132, it follows that there are three ways of reducing the quartic to the proposed form, the determination of which ultimately depends on the solution of the reducing cubic of the quartic.

194. **Removal of the Second, Third, and Fourth Terms from an Equation of the n^{th} Degree.**—We begin by proving the following proposition, which we shall subsequently apply:—

A homogeneous function V of the second degree in n quantities $x_1, x_2, x_3, \ldots x_n$ can be expressed in general as the sum of n squares.

To prove this, let V, arranged in powers of x_1, take the following form:—

$$V \equiv P_1x_1^2 + 2Q_1x_1 + R_1,$$

where P_1 does not contain $x_1, x_2, \ldots . x_n$; also Q_1 and R_1 are linear and quadratic functions, respectively, of $x_2, x_3, \ldots x_n$. We have then

$$V \equiv \left(\sqrt{P_1}x_1 + \frac{Q_1}{\sqrt{P_1}}\right)^2 + R_1 - \frac{Q_1^2}{P_1};$$

also, assuming

$$V_1 \equiv R_1 - \frac{Q_1^2}{P_1} = P_2x_2^2 + 2Q_2x_2 + R_2,$$

where P_2 is a constant, and Q_2 and R_2 do not contain x_1 and x_2, we have, similarly,

$$V_1 = \left(\sqrt{P_2}x_2 + \frac{Q_2}{\sqrt{P_2}}\right)^2 + R_2 - \frac{Q_2^2}{P_2};$$

so that

$$V = \left(\sqrt{P_1}x_1 + \frac{Q_1}{\sqrt{P_1}}\right)^2 + \left(\sqrt{P_2}x_2 + \frac{Q_2}{\sqrt{P_2}}\right)^2 + R_2 - \frac{Q_2^2}{P_2}.$$

Proceeding in this way, we arrive ultimately at $R_{n-1} - \dfrac{Q^2_{n-1}}{P_{n-1}}$, which is equal to $P_nx_n^2$; and the proposition is proved.

Now, returning to the original problem, let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0;$$

and, putting

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon,$$

let the transformed equation be

$$y^n + Q_1y^{n-1} + Q_2y^{n-2} + \ldots + Q_n = 0,$$

where, by Art. 188, $Q_1, Q_2, \ldots Q_r, \ldots$ are homogeneous functions of the first, second, ... r^{th} degrees in $\alpha, \beta, \gamma, \delta, \epsilon$.

Now, if $\alpha, \beta, \gamma, \delta, \epsilon$ can be determined so that

$$Q_1 = 0, \quad Q_2 = 0, \ Q_3 = 0,$$

the problem will be solved. For this purpose, eliminating ϵ from Q_2 and Q_3, by substituting its value derived from $Q_1 = 0$, we obtain two homogeneous equations

$$R_2 = 0, \quad R_3 = 0,$$

of the second and third degrees in $\alpha, \beta, \gamma, \delta$; and by the proposition proved above we may write R_2 under the form

$$u^2 - v^2 + w^2 - t^2,$$

which is satisfied by putting $u = v$ and $w = t$. From these simple equations we find $\gamma = l\alpha + m\beta$, and $\delta = l_1\alpha + m_1\beta$; and substituting these values in $Q_3 = 0$, we have a cubic equation to determine the ratio $\beta : \alpha$. Whence, giving any one of the quantities $\alpha, \beta, \gamma, \delta, \epsilon$ a definite value, the rest are determined, and the equation is reduced to the form

$$y^n + Q_4 y^{n-4} + Q_5 y^{n-5} + \ldots + Q_n = 0.$$

In a similar way we may remove the coefficients Q_1, Q_2, Q_4, by solving an equation of the fourth degree.

Applying this method to the quintic, we may reduce it to either of the trinomial forms*

$$x^5 + Px + Q, \quad x^5 + Px^2 + Q;$$

or again, changing x into $\frac{1}{x}$, to either of the forms

$$x^5 + Px^3 + Q, \quad x^5 + Px^4 + Q.$$

In this investigation we have followed M. Serret (see his *Cours d' Algèbre Supérieure*, Vol. I., Art 192).

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* See Note A.

Section II.—Hermite's Theorem.

195. **Homogeneous Function of Second Degree expressed as Sum of Squares.**—We have already shown, in a general way (Art. 194), that a homogeneous function of the second degree in the variables may be reduced to a sum of squares, no hypothesis being made as to the nature of the coefficients of the function considered. We now return to the consideration of this problem when the coefficients of the function are supposed to be all *real;* and we proceed to determine, in magnitude and sign, the coefficients of the squares in the transformed function.

Let $F(x_1, x_2, \ldots x_n)$ be a homogeneous function of the second degree in n variables with real coefficients; and let us suppose that it is reduced by the method of Art. 194 to the form

$$\begin{aligned} p_1(x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n)^2 & \\ + p_2(x_2 + b_3x_3 + \ldots + b_nx_n)^2 & \\ + p_3(x_3 + \ldots + c_nx_n)^2 & \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ + p_nx_n^2, & \end{aligned}$$

where all the coefficients of this new form are real.

Making now the linear substitution

$$\begin{aligned} X_1 &= x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n, \\ X_2 &= \qquad\quad x_2 + b_3x_3 + b_4x_4 + \ldots + b_nx_n, \\ X_3 &= \qquad\qquad\qquad x_3 + c_4x_4 + \ldots + c_nx_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_n &= \qquad\qquad\qquad\qquad\qquad\qquad\quad x_n, \end{aligned}$$

we have

$$F(x_1, x_2, x_3, \ldots x_n) \equiv p_1X_1^2 + p_2X_2^2 + p_3X_3^2 + \ldots + p_nX_n^2.$$

Since the modulus of this transformation is equal to 1, the discriminants of both these forms of F must be absolutely equal.

Denoting, therefore, the discriminant of F by Δ_n, we have

$$\Delta_n = p_1 p_2 p_3 \dots p_n ;$$

and similarly, when the variables $x_{j+1}, x_{j+2}, \dots x_n$ are made to vanish in both forms of F, we have

$$\Delta_j = p_1 p_2 p_3 \dots p_j.$$

Now, giving j the values 1, 2, 3, &c., we find

$$p_1 = \Delta_1, \quad p_2 = \frac{\Delta_2}{\Delta_1}, \quad p_3 = \frac{\Delta_3}{\Delta_2}, \dots p_n = \frac{\Delta_n}{\Delta_{n-1}} ;$$

and the coefficients are determined in terms of the discriminant of the original quadratic form in n variables and the discriminants of the forms in $n - 1$, $n - 2$, &c., variables derived from the given form by causing one, two, &c., of the variables to vanish in succession in the manner just explained.

Again since the constants in the form $F(x_1, x_2, \dots x_n)$ are in number $\frac{1}{2}n(n-1)$ less than in a form composed of a sum of squares of n linear functions of n variables, we learn that F can be reduced to a sum of squares in an infinity of ways. It is most important, however, to observe that *in whatever way the transformation is made, provided it is real, the number of coefficients (affecting these squares) which have a given sign is always the same.* This theorem, which is due to Jacobi, is easily proved; for suppose the contrary possible, and let

$$F = p_1 X_1^2 + p_2 X_2^2 + \dots + p_n X_n^2 = q_1 Y_1^2 + q_2 Y_2^2 + \dots + q_n Y_n^2,$$

where the number of positive coefficients on both sides of this identity is not the same. Making all the terms positive, by transferring those affected with negative signs to the opposite sides of the identity, we shall have a sum of l squares identically equal to a sum of m squares, where m is greater than l. Now, substituting such values for $x_1, x_2, \dots x_n$ that each of the l squares may vanish (which may be done in an infinity of ways), we find a sum of m squares identically equal to zero, which is impossible.

196. **Hermite's Theorem.**—The principles explained in the preceding Article have been applied by Hermite to the determination of the number of real roots of an equation $f(x)=0$ comprised within given limits. The special form of the function F which he makes use of for this purpose is

$$\sum_{r=1}^{r=n} \frac{1}{a_r - \rho} (x_1 + a_r x_2 + a_r^2 x_3 + \ldots + a_r^{n-1} x_n)^2,$$

in which $x_1, x_2, \ldots x_n$ are any variables in number equal to the degree of the equation; and r takes all values from 1 to n inclusive, the roots of the equation being $a_1, a_2, \ldots a_n$; also ρ is any arbitrary parameter.

This form is plainly a symmetric function of the roots of the equation $f(x)=0$; and as the coefficients of this equation are supposed to be real, F will be also real, when expressed in terms of these coefficients and ρ, provided the parameter ρ be given any real value. If the roots $a_1, a_2, a_3, \ldots a_n$ are not all real, the assumed form of F will not be obtained by real transformation; but it is easy to deduce from it, as follows, another form which will be so obtained.

If a_1 and a_2 be a pair of conjugate imaginary roots, we may write

$$a_1 = r_0(\cos a + i \sin a), \quad a_2 = r_0(\cos a - i \sin a).$$

Denoting for shortness $x_1 + a_r x_2 + a_r^2 x_3 + \ldots + a_r^{n-1} x_n$ by Y_r, and substituting these values in Y_1 and Y_2, we find

$$Y_1 \equiv U + iV, \quad Y_2 \equiv U - iV,$$

where U and V are real; also putting

$$\frac{1}{a_1 - \rho} = r(\cos\phi + i\sin\phi), \quad \frac{1}{a_2 - \rho} = r(\cos\phi - i\sin\phi),$$

the part of the function F depending on a_1 and a_2, viz.,

$$\frac{Y_1^2}{a_1 - \rho} + \frac{Y_2^2}{a_2 - \rho},$$

becomes

$$r\left\{\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)^2(U+iV)^2+\left(\cos\frac{\phi}{2}-i\sin\frac{\phi}{2}\right)^2(U-iV)^2\right\},$$

which may be also written as the difference of the squares

$$2r\left(U\cos\frac{\phi}{2}-V\sin\frac{\phi}{2}\right)^2-2r\left(U\sin\frac{\phi}{2}+V\cos\frac{\phi}{2}\right)^2;$$

proving that two imaginary conjugate roots introduce into F two real squares, one of which has a positive and the other a negative coefficient.

We now state Hermite's theorem as follows:—*Let the equation* $f(x)=(x-a_1)(x-a_2)\ .\ .\ .\ (x-a_n)=0$ *have real coefficients and unequal roots: if then by a* REAL *substitution we reduce*

$$\frac{Y_1^2}{a_1-\rho}+\frac{Y_2^2}{a_2-\rho}+\frac{Y_3^2}{a_3-\rho}+\ .\ .\ .+\frac{Y_n^2}{a_n-\rho}, \qquad (1)$$

where $\qquad Y_r=x_1+a_rx_2+a_r^2x_3+\ .\ .\ .\ +a_r^{n-1}x_n,$

to a sum of squares, the number of squares having positive coefficients will be equal to the number of pairs of imaginary roots of the equation $f(x)=0$, *augmented by the number of real roots greater than* ρ.

This theorem follows at once from what has preceded if we consider separately the parts of the function (1) which refer to real roots and to imaginary roots, for obviously there is a positive square for every root greater than ρ, and we have proved that every pair of conjugate imaginary roots leads to a positive and negative real square, without affecting the other squares independent of these roots.

The number of real roots between any two numbers ρ_1 and ρ_2 may be readily estimated. For, denoting in general by P_j the number of positive squares in F when $\rho=\rho_j$, by N_j the number of roots of the equation $f(x)=0$ greater than ρ_j, and by $2I$ the number of imaginary roots, we have

$$P_1=N_1+I,\quad P_2=N_2+I;$$

whence

$$N_1-N_2=P_1-P_2,$$

proving that the number of real roots between ρ_1 and ρ_2 is equal to the difference between the number of positive squares when ρ has the values ρ_1 and ρ_2, respectively.

The number here determined may be shown to depend on a very important series of functions connected with the given equation. In order to derive these functions we consider F under the form (Art. 195)

$$\Delta_1 X_1^2 + \frac{\Delta_2}{\Delta_1} X_2^2 + \frac{\Delta_3}{\Delta_2} X_3^2 + \ldots + \frac{\Delta_n}{\Delta_{n-1}} X_n^2.$$

The number P expresses the number of coefficients in this form which are positive, or, which is the same thing, the number of the following quantities which are negative :—

$$-\frac{\Delta_1}{1}, \; -\frac{\Delta_2}{\Delta_1}, \; -\frac{\Delta_3}{\Delta_2}, \ldots - \frac{\Delta_n}{\Delta_{n-1}}. \qquad (2)$$

We proceed now to calculate $\Delta_1, \Delta_2, \ldots \Delta_j, \ldots \Delta_n$ in terms of ρ and the roots of the equation $f(x) = 0$; and as the method is the same in every case it will be sufficient to calculate Δ_3, i. e. the discriminant of the original form of F when all the variables except x_1, x_2, x_3 vanish.

Writing for shortness $\nu_r = \dfrac{1}{a_r - \rho}$, we have in this case

$$F_3 = \Sigma\nu_r (x_1 + a_r x_2 + a_r^2 x_3)^2.$$

The discriminant in this form is

$$\Delta_3 = \begin{vmatrix} \Sigma\nu & \Sigma a\nu & \Sigma a^2\nu \\ \Sigma a\nu & \Sigma a^2\nu & \Sigma a^3\nu \\ \Sigma a^2\nu & \Sigma a^3\nu & \Sigma a^4\nu \end{vmatrix},$$

which may be written as the product of the two arrays

$$\left.\begin{matrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \end{matrix}\right\}, \qquad \left\{\begin{matrix} \nu_1 & \nu_2 & \ldots & \nu_n \\ a_1\nu_1 & a_2\nu_2 & \ldots & a_n\nu_n \\ a_1^2\nu_1 & a_2^2\nu_2 & \ldots & a_n^2\nu_n \end{matrix}\right\};$$

and, consequently,

$$\Delta_3 = \Sigma\nu_1\nu_2\nu_3 \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix}^2 = \sum \frac{(a_2 - a_3)^2 (a_3 - a_1)^2 (a_1 - a_2)^2}{(a_1 - \rho)(a_2 - \rho)(a_3 - \rho)}.$$

In an exactly similar manner we find

$$\Delta_j = \sum \frac{\nabla (a_1, a_2, a_3 \ldots a_j)}{(a_1 - \rho)(a_2 - \rho) \ldots (a_j - \rho)},$$

where the notation $\nabla (a_1, a_2, a_3, \ldots a_j)$ is employed to represent the product of the squares of the differences of $a_1, a_2, a_3, \ldots a_j$. Hence the quantities $\Delta_1, \Delta_2, \Delta_j \ldots \Delta_n$ are all determined.

Now, multiplying the numerator and denominator of each of the fractions in the series (2) by $f(\rho)$, each value of Δ is rendered integral, and the series becomes

$$\frac{V_1}{V}, \quad \frac{V_2}{V_1}, \quad \frac{V_3}{V_2}, \ldots \frac{V_n}{V_{n-1}},$$

where

$$V = (\rho - a_1)(\rho - a_2) \ldots . (\rho - a_n),$$

$$V_1 = \Sigma (\rho - a_2)(\rho - a_3) \ldots . (\rho - a_n),$$

$$V_2 = \Sigma \nabla (a_1, a_2)(\rho - a_3) \ldots . (\rho - a_n),$$

$$V_3 = \Sigma \nabla (a_1, a_2, a_3)(\rho - a_4) \ldots . (\rho - a_n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$V_n = \nabla (a_1, a_2, a_3, \ldots . a_n).$$

Since negative terms in the series (3) correspond to variations of sign in the series $V, V_1, V_2, V_3, \ldots . V_n$, it is proved that the number of variations lost in the series last written, when ρ passes from the value ρ_1 to the value ρ_2, is exactly equal to the number of real roots of the equation $f(\rho) = 0$ comprised between ρ_1 and ρ_2.

197. **Sylvester's Forms of Sturm's Functions.**—It will be observed that the functions V, V_1, V_2, &c., arrived at in the preceding Article have the same property as Sturm's

functions; from which in fact they differ by positive multipliers only, as was observed by Sylvester, who first published these forms in the *Philosophical Magazine*, December, 1839. The identity of the two series of functions may be established as follows:—

We make use of the notation already employed in Art. 148, and we propose to show that the Sturmian remainder R_j differs only by the positive factor γ_j from the function V_j. From the Article referred to, we have

$$R_j \equiv A_j f'(x) - B_j f(x), \tag{1}$$

where
$$R_j = r_0 + r_1 x + r_2 x^2 + \ldots + r_{n-j} x^{n-j},$$
$$A_j = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{j-1} x^{j-1},$$
$$B_j = \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_{j-2} x^{j-2};$$

and from the value of r_{n-j} there given we have immediately

$$r_{n-j} = \gamma_j \Sigma \nabla (a_1, a_2, a_3, \ldots a_j),$$

showing that the leading coefficients in R_j and V_j differ only by the factor γ_j. We now proceed to prove that the last coefficients in these functions differ only by the same factor. For this purpose, dividing the identity (1) by $f(x)$, substituting in it from the equation

$$\frac{f'(x)}{f(x)} = \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots,$$

and comparing coefficients, we find

$$\begin{aligned}
\mu_0 &= \lambda_1 s_0 + \lambda_2 s_1 + \lambda_3 s_2 + \ldots + \lambda_{j-1} s_{j-2}, \\
\mu_1 &= \qquad\quad \lambda_2 s_0 + \lambda_3 s_1 + \ldots + \lambda_{j-1} s_{j-3}, \\
&\ldots\ldots\ldots\ldots\ldots \\
\mu_{j-2} &= \qquad\qquad\qquad\qquad\qquad \lambda_{j-1} s_0.
\end{aligned}$$

Also, putting $x = 0$ in (1), we have

$$r_0 = \lambda_0 p_{n-1} - \mu_0 p_n,$$

and, substituting for μ_0 in terms of $\lambda_1, \lambda_2, \lambda_3$, &c.,

$$-\frac{r_0}{p_n} = \lambda_0 s_{-1} + \lambda_1 s_0 + \lambda_2 s_1 + \ldots + \lambda_{j-1} s_{j-2};$$

whence, giving to $\lambda_0, \lambda_1, \ldots \lambda_{j-1}$ the same values as in the calculation of r_{n-j}, we find

$$r_0 = (-1)^j p_n \gamma_j \begin{vmatrix} s_{-1} & s_0 & s_1 \ldots s_{j-2} \\ s_0 & s_1 & s_2 \ldots s_{j-1} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ s_{j-2} & s_{j-1} & s_j \ldots s_{2j-3} \end{vmatrix}.$$

Now, referring to the calculation of Δ_j in Art. 196, and putting $\rho = 0$, or $\nu_r = \dfrac{1}{a_r}$, in the value of Δ_j there found, we find for the determinant just written the value

$$\sum \frac{\nabla(a_1, a_2, a_3, \ldots a_j)}{a_1 a_2 a_3 \ldots a_j};$$

hence, giving p_n its value in terms of the roots, we have

$$r_0 = (-1)^{n-j} \gamma_j \Sigma \nabla (a_1, a_2, a_3, \ldots a_j)\, a_{j+1} a_{j+2} \ldots a_n,$$

which was required to be proved.

The first and last coefficients of R_j, when divided by γ_j, having been thus shown to be the same as in the form V_j, it follows that all the intermediate terms must be similarly related; for, in the first place, R_j is a function of the differences of the quantities $x, a_1, a_2 \ldots a_n$, as may be seen by transforming $f(x)$ before calculating R_j by the substitution $z = a_0 x + a_1$, as in Ex. 3, Art. 92. When this transformation is completed, every coefficient in R_j, as well as z, is a function of the differences; consequently R_j is a semicovariant, and satisfies the differential equation

$$\left(\frac{d}{dx} + \frac{d}{da_1} + \frac{d}{da_2} + \ldots + \frac{d}{da_n}\right) R_j = 0, \text{ or } \frac{dR_j}{dx} - DR_j = 0,$$

and therefore, as is proved in Article 147, all the coefficients may be obtained from the last by a definite law. The same conclusions plainly holding also for the function V_j, it is therefore proved, finally, that

$$R_j = \gamma_j V_j.$$

Examples.

1. To reduce two quadrics in three variables to the sums of the same three squares with proper coefficients.

Let

$$U = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

$$V = a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy,$$

$$F(x, y, z) = \lambda U + V, \quad X = \tfrac{1}{2}\frac{dF}{dx}, \quad Y = \tfrac{1}{2}\frac{dF}{dy}, \quad Z = \tfrac{1}{2}\frac{dF}{dz}.$$

We have then identically

$$F = -\frac{1}{\Delta(\lambda)}\begin{vmatrix} \lambda a + a_1 & \lambda h + h_1 & \lambda g + g_1 & X \\ \lambda h + h_1 & \lambda b + b_1 & \lambda f + f_1 & Y \\ \lambda g + g_1 & \lambda f + f_1 & \lambda c + c_1 & Z \\ X & Y & Z & 0 \end{vmatrix} = \frac{\Phi(\lambda)}{\Delta(\lambda)},$$

where $\Delta(\lambda)$ is the discriminant of $\lambda U + V$; and $\Phi(\lambda)$ is a function of the 2nd degree in λ, the symbols X, Y, Z being retained in it for the present, and not replaced by the values involving λ.

Resolving into partial fractions, we have

$$F = \frac{\Phi(\lambda_1)}{\Delta'(\lambda_1)}\frac{1}{\lambda - \lambda_1} + \frac{\Phi(\lambda_2)}{\Delta'(\lambda_2)}\frac{1}{\lambda - \lambda_2} + \frac{\Phi(\lambda_3)}{\Delta'(\lambda_3)}\frac{1}{\lambda - \lambda_3}, \tag{1}$$

in which $\Phi(\lambda_1)$, $\Phi(\lambda_2)$, $\Phi(\lambda_3)$ are all perfect squares, since they are obtained by bordering the vanishing determinants $\Delta(\lambda_1)$, $\Delta(\lambda_2)$, $\Delta(\lambda_3)$. (See Art. 139.)

Now, replacing X, Y, Z by their values, $\lambda U_1 + V_1$, &c., $\Phi(\lambda_j)$ is easily reducible to the form

$$-(\lambda - \lambda_j)^2\begin{vmatrix} \lambda_j a + a_1 & \lambda_j h + h_1 & \lambda_j g + g_1 & U_1 \\ \lambda_j h + h_1 & \lambda_j b + b_1 & \lambda_j f + f_1 & U_2 \\ \lambda_j g + g_1 & \lambda_j f + f_1 & \lambda_j c + c_1 & U_3 \\ U_1 & U_2 & U_3 & 0 \end{vmatrix} \equiv (\lambda - \lambda_j)^2 u_j^2,$$

where $j = 1, 2,$ or 3, and u_j is independent of λ.

Substituting these values in (1), we find

$$\lambda U + V \equiv (\lambda - \lambda_1)\frac{u_1^2}{\Delta'(\lambda_1)} + (\lambda - \lambda_2)\frac{u_2^2}{\Delta'(\lambda_2)} + (\lambda - \lambda_3)\frac{u_3^2}{\Delta'(\lambda_3)}.$$

Equating the coefficients of λ, we have

$$U = \frac{u_1^2}{\Delta'(\lambda_1)} + \frac{u_2^2}{\Delta'(\lambda_2)} + \frac{u_3^2}{\Delta'(\lambda_3)},$$

$$-V = \lambda_1 \frac{u_1^2}{\Delta'(\lambda_1)} + \lambda_2 \frac{u_2^2}{\Delta'(\lambda_2)} + \lambda_3 \frac{u_3^2}{\Delta'(\lambda_3)},$$

which was required to be done.

It is to be observed that this problem has only one solution. The mode of reduction here given is due to Darboux; and is plainly applicable whatever be the number of variables.

2. Prove that a quadric in n variables may be reduced by a *real* orthogonal transformation to a sum of n squares.

An orthogonal transformation is a linear transformation such that, when the modulus written as a determinant is squared the terms in the principal diagonal are each equal to 1, and all the other terms vanish.

In a transformation of this kind it follows that the sum of the squares of the new variables is equal to the sum of the squares of the old.

3. Writing as before one of Sturm's remainders in the form

$$R_j \equiv A_j\phi'(x) - B_j\phi(x),$$

prove that

$$B_j = \gamma_j \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{j-1} \\ s_1 & s_2 & s_3 & \cdots & s_j \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ s_{j-2} & s_{j-1} & s_j & \cdots & s_{2j-3} \\ 0 & T_1 & T_2 & \cdots & T_{j-1} \end{vmatrix},$$

where $$T_j = s_0x^{j-1} + s_1x^{j-2} + s_2x^{j-3} + \ldots + s_{j-1}.$$

4. Denoting by U_n

$$\sum_{r=1}^{r=n} (x - a_r)(x_1 + ax_2 + a^2x_3 + \ldots + a^{n-1}x_n)^2,$$

prove that the discriminant of U_j may be determined by the equation

$$\Delta_j = \frac{A_j}{\gamma_j},$$

where A_j and γ_j have the same signification as before; and show directly that if $\Delta_j = 0$ for a certain value of x, Δ_{j-1} and Δ_{j+1} have opposite signs for the same value of x.

NOTE.—Hermite's theorem holds where $a_r - \rho$ is changed into $(a_r - \rho)^m$ in the enunciation on p. 435, m being any odd integer, positive or negative.

Section III.—Miscellaneous Theorems.

198. **Reduction of the Quintic to the Sum of Three Fifth Powers.**—This reduction can be effected by the solution of an equation of the third degree, as we proceed to show. Let

$$(a_0, a_1, a_2, a_3, a_4, a_5)(x, y)^5 \equiv b_1(x+\beta_1 y)^5 + b_2(x+\beta_2 y)^5 + b_3(x+\beta_3 y)^5,$$

where β_1, β_2, β_3 are the roots of the equation

$$p_3 x^3 + p_2 x^2 + p_1 x + p_0 = 0.$$

Now, comparing coefficients in the two forms of the quintic,

$$a_0 = b_1 \quad + b_2 \quad + b_3 \quad, \quad a_1 = b_1\beta_1 + b_2\beta_2 + b_3\beta_3,$$
$$a_2 = b_1\beta_1^2 + b_2\beta_2^2 + b_3\beta_3^2, \quad a_3 = b_1\beta_1^3 + b_2\beta_2^3 + b_3\beta_3^3,$$
$$a_4 = b_1\beta_1^4 + b_2\beta_2^4 + b_3\beta_3^4, \quad a_5 = b_1\beta_1^5 + b_2\beta_2^5 + b_3\beta_3^5;$$

whence

$$p_0 a_0 + p_1 a_1 + p_2 a_2 + p_3 a_3 = 0,$$
$$p_0 a_1 + p_1 a_2 + p_2 a_3 + p_3 a_4 = 0,$$
$$p_0 a_2 + p_1 a_3 + p_2 a_4 + p_3 a_5 = 0.$$

When these equations are taken in conjunction with the equation

$$p_0 + p_1 x + p_2 x^2 + p_3 x^3 = 0,$$

we have the following equation to determine β_1, β_3, β_3:—

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

Also, b_1, b_2, b_3 are determined by the equations

$$b_1 \quad + b_2 \quad + b_3 \quad = a_0,$$
$$b_1\beta_1 + b_2\beta_2 + b_3\beta_3 = a_1,$$
$$b_1\beta_1^2 + b_2\beta_2^2 + b_3\beta_3^2 = a_2;$$

whence the question is completely solved when β_1, β_2, β_3 are known.

This important transformation of the quintic is a particular case of the following general theorem due to Sylvester :—

Any homogeneous function of x, y, of the degree $2n-1$, can be reduced to the form

$$b_1(x+\beta_1 y)^{2n-1} + b_2(x+\beta_2 y)^{2n-1} + \ldots + b_n(x+\beta_n y)^{2n-1}$$

by the solution of an equation of the n^{th} degree.

The proof of the general theorem is exactly similar to that above given for the case of the quintic.

199. **Quartics Transformable into each other.**—We proceed to determine under what conditions two quartics can be transformed, the one into the other, by linear transformation.

Let the quartics be

$$U = (a, b, c, d, e)(x, y)^4 = a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y),$$

$$V = (a', b', c', d', e')(x', y')^4 = a'(x'-\alpha' y')(x'-\beta' y')(x'-\gamma' y')(x'-\delta' y');$$

and if they become identical by the transformation

$$x' = \lambda x + \mu y, \quad y' = \lambda' x + \mu' y,$$

we have, by Art. 38,

$$\frac{(\beta'-\gamma')(\alpha'-\delta')}{(\beta-\gamma)(\alpha-\delta)} = \frac{(\gamma'-\alpha')(\beta'-\delta')}{(\gamma-\alpha)(\beta-\delta)} = \frac{(\alpha'-\beta')(\gamma'-\delta')}{(\alpha-\beta)(\gamma-\delta)},$$

showing that the six anharmonic ratios determined by the roots must be the same for both equations.

From these equations we have also the following relations between the invariants of the two forms :—

$$I' = r^4 I, \quad J' = r^6 J; \tag{1}$$

whence

$$\frac{I'^3}{J'^2} = \frac{I^3}{J^2}. \tag{2}$$

The quantity $\frac{I^3}{J^2}$ being absolutely unaltered by transformation when the quartic is linearly transformed, is called the *absolute invariant* of the quartic. The condition expressed by equation (2) is, therefore, that the absolute invariant should be the same for both quartics. The condition here arrived at agrees with the result of Ex. 16, p. 148, where it is proved that the sextic which determines the anharmonic ratios of the roots involves the absolute invariant, and no other function of the coefficients of the quartic.

The conditions expressed by the equations (1), (2), are always *necessary;* but not always *sufficient*, as we proceed to illustrate by two exceptional cases.

Suppose, in the first place,

$$U \equiv u^2vw, \quad V \equiv u'^2v'^2,$$

where u, v, w, u', v', are of the linear form $lx + my$.

Although the condition $\frac{I^3}{J^2} = \frac{I'^3}{J'^2}$ is satisfied in this case, the common value of these fractions being 27, it is impossible to transform U into V, since it is impossible to make vw a perfect square by linear transformation.

Secondly, if $$U \equiv u^3v, \quad V \equiv u'^4;$$

although the equations $I' = r^4I$, $J' = r^6J$ are satisfied, since $I' = 0$, $I = 0$, $J' = 0$, $J = 0$, it is nevertheless impossible to transform U into V.

In both these cases it would be impossible to identify the six anharmonic ratios depending on the roots of the quartics. In general, it may be stated that it is impossible to transform one quantic into another by linear transformation when any relation exists between the invariants of one of them which does not exist between the invariants of the other (see Clebsch's *Theorie der Binären Algebraischen Formen*, Art. 92).

200. **Number of Absolute Invariants of any Quantic.**—We proceed now to examine how the number of absolute invariants of any binary quantic is connected with the number of ordinary invariants, and how far a limit can be determined to either of these numbers. Transforming the quantic

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

by the substitution

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y;$$

if the new form be

$$(A_0, A_1, A_2 \ldots A_n)(X, Y)^n,$$

we have by the comparison of coefficients $n + 1$ equations expressing $A_0, A_1 \ldots A_n$ as follows:—

$$A_0 = (a_0, a_1, a_2 \ldots a_n)(\lambda, \lambda')^n, \ldots A_j = \frac{\Gamma_{(j)}}{\Gamma_{(n)}} \Delta^{n-j} A_n, \ldots$$

$$A_n = (a_0, a_1, a_2, \ldots a_n)(\mu, \mu')^n,$$

where

$$\Delta = \lambda \frac{d}{d\mu} + \lambda' \frac{d}{d\mu'}, \quad \Gamma_{(j)} = 1 \cdot 2 \cdot 3 \ldots j, \quad \Gamma_{(o)} = 1.$$

Now, eliminating $\lambda, \mu, \lambda', \mu'$, we obtain, among the new and old coefficients, $n - 3$ independent relations; but if $(\lambda\mu' - \lambda'\mu)$ be admitted when $\lambda, \mu, \lambda', \mu'$ are excluded by elimination, we must add the equation $\lambda\mu' - \lambda'\mu = M$ to the $n + 1$ equations already obtained, making $n + 2$ in all; and when the elimination is now completed, we have $n - 2$ independent relations. It may be inferred from our previous investigations that these relations are of the form

$\phi_r(A_0, A_1, A_2, \ldots A_n) = M^j \phi_r(a_0, a_1, a_2 \ldots a_n)$ (see Art. 164),

and we have therefore $n - 2$ independent ordinary invariants $\phi_1, \phi_2, \phi_3 \ldots \phi_{n-2}$. Eliminating M we obtain, as above stated, $n - 3$ relations connecting the two sets of coefficients, and this, therefore, is the number of independent absolute invariants. It is not true in general that every invariant can be expressed as a *rational* function of the invariants $\phi_1, \phi_2, \phi_3 \ldots \phi_{n-2}$; and, consequently, we have not obtained a superior limit to the number of independent ordinary invariants by this investigation.

201. **Number of Seminvariants of a Quantic.**—Every seminvariant can be expressed rationally in terms of a_0 and $n-1$ functions of the coefficients which are either invariants or seminvariants. For, removing the second term from the equation

$$U_n \equiv (a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

the new coefficients are easily obtained by substituting for h its value $-\frac{a_1}{a_0}$ (see Art. 35). As these coefficients, when divided by a_0, are symmetric functions of the differences of the roots they must be invariants or seminvariants when multiplied by a power of a_0; also every other symmetric function of the differences of the roots must be a rational function of the same quantities, but not necessarily *integral* when multiplied by a_0^{ϖ}; consequently we have not obtained any superior limit to the number of independent seminvariants (or, which is the same thing, covariants) by this investigation.

As an illustration of the preceding we give the values of A_2, A_3, A_4, A_5, A_6 in a reduced form—

$$a_0 A_2 = H, \quad a_0^2 A_3 = G, \quad a_0^3 A_4 = a_0^2 I - 3H^2 \text{ (see Art. 37)},$$

$$a_0^4 A_5 = a_0^2 F - 2GH,$$

$$a_0^5 A_6 = 45H^3 - 15a_0^2 HI + 10G^2 + a_0^4 I_2,$$

where

$$F = a_0^2 a_5 - 5a_0 a_1 a_4 + 2a_0 a_2 a_3 - 6a_1 a_2^2 + 8a_1^2 a_3,$$

$$I_2 = a_0 a_6 - 6a_1 a_5 + 15a_2 a_4 - 10a_3^2,$$

F being a seminvariant, and I_2 an invariant of the sextic U_6. We have, therefore, proved that every seminvariant of the sextic can be expressed in the form

$$a_0^{-r}\,\Psi(a_0, F, G, H, I, I_2),$$

where Ψ is a rational and integral function; and, consequently, every covariant when multiplied by a power of U_6 may be expressed as follows:—

$$\Psi(U_6, F_x, G_x, H_x, I_x, I_2).$$

When a rational and integral function of several seminvariants is formed so that the result is divisible by a_0, a new seminvariant is obtained which is considered distinct from the others.

202. **Galois' Theorem.**—*The roots of an equation*

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n) = 0$$

can be determined when one of the values is given of a rational function of those roots which has $1 \cdot 2 \cdot 3 \dots n$ *distinct values when they are permuted in every way.*

Let $V = \phi(a_1, a_2, a_3 \dots a_n)$ be this rational function, and V_1 the given value of it. Now, permuting all the roots except a_1, we have $1 \cdot 2 \cdot 3 \dots n - 1 = \mu$ values of V given by the equation

$$(V - V_1)(V - V_2)(V - V_3) \dots (V - V_\mu) = 0. \qquad (1)$$

This equation can be put under the form

$$F(V, a_1) = 0,$$

for its coefficients are symmetric functions of the roots of the equation

$$\frac{f(x)}{x - a_1} = 0,$$

and as (1) is satisfied by $V = V_1$, we have identically $F(V_1, a_1) = 0$. Hence $f(x) = 0$ and $F(V_1, x) = 0$ have one common root, and only one. If, therefore, we seek the common measure of $f(x)$ and $F(V_1, x)$, and continue the process till we obtain a remainder of the first degree in x, and equate it to zero, we shall find $a_1 = \Psi_1(V_1)$. Fixing on another root a_r, and proceeding as before, we shall find $a_r = \Psi_r(V_1)$; and therefore each root can be determined rationally in terms of the given value of V.

203. **Reciprocal Linear Transformation.**—When the co-ordinates of a point are transformed by a linear transformation, the tangential co-ordinates of a line and the operating symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ are both transformed by the same new linear transformation, which is said to be reciprocal to the first.

Let the linear transformation be

$$\begin{aligned} x &= a_1X + b_1Y + c_1Z, \\ y &= a_2X + b_2Y + c_2Z, \\ z &= a_3X + b_3Y + c_3Z; \end{aligned} \qquad (1)$$

whence any line $\lambda x + \mu y + \nu z$ becomes by transformation $LX + MY + NZ$, where

$$\begin{aligned} L &= a_1\lambda + a_2\mu + a_3\nu, \\ M &= b_1\lambda + b_2\mu + b_3\nu. \\ N &= c_1\lambda + c_2\mu + c_3\nu\,; \end{aligned} \tag{2}$$

also

$$\frac{d}{dX} = \frac{d}{dx}\,\frac{dx}{dX} + \frac{d}{dy}\,\frac{dy}{dX} + \frac{d}{dz}\,\frac{dz}{dX},$$

or, substituting for $\frac{dx}{dX}, \frac{dy}{dX}, \frac{dz}{dX}$ their values,

$$\frac{d}{dX} = a_1\frac{d}{dx} + a_2\frac{d}{dy} + a_3\frac{d}{dz},$$

and similarly

$$\frac{d}{dY} = b_1\frac{d}{dx} + b_2\frac{d}{dy} + b_3\frac{d}{dz}, \quad \frac{d}{dZ} = c_1\frac{d}{dx} + c_2\frac{d}{dy} + c_3\frac{d}{dz}\,;$$

whence L, M, N and the symbols $\frac{d}{dX}, \frac{d}{dY}, \frac{d}{dZ}$ follow the same laws of transformation, and consequently λ, μ, ν and $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ also; in fact from equations (2) this transformation is

$$\begin{aligned} \Delta\lambda &= A_1L + B_1M + C_1N, \\ \Delta\mu &= A_2L + B_2M + C_2N, \\ \Delta\nu &= A_3L + B_3M + C_3N, \end{aligned}$$

where $\quad \Delta = (a_1b_2c_3), \quad A_1 = \frac{d\Delta}{da_1}, \quad B_1 = \frac{d\Delta}{db_1}$, &c. &c.

This linear transformation is said to be reciprocal to the transformation (1) whose modulus is Δ, its coefficients being

$$\frac{1}{\Delta}\frac{d\Delta}{da_1}, \ \frac{1}{\Delta}\frac{d\Delta}{db_1}, \ \frac{1}{\Delta}\frac{d\Delta}{dc_1}, \text{ \&c.}$$

The variables x, y, z, and $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ are said to be *contragredient* to each other, for a linear transformation of x, y, z leads to a linear transformation of the symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, which, although not the same, is connected with the first in the manner already explained.

Miscellaneous Examples.

1. Every quantic of an odd degree has a quadratic covariant of the second order in the coefficients.

For every quantic of an even degree has an invariant of the second order in the coefficients (Art. 171), which may be written in the form $U_D(U)$ or $(1, 2)^n U_1 U_2$; and this invariant of the quantic whose degree is $2m$ will be a seminvariant of one whose degree is $2m + 1 = n$. The covariant therefore which has this seminvariant as leader will be a quadratic, since $n\varpi - 2\kappa = 2$, κ being $= n - 1$ and $\varpi = 2$.

2. Every quantic of an odd degree $2m + 1 = n$ has a linear covariant of the degree n in the coefficients when n is greater than 3.

For if $I(x, y)^2$ be the quadratic covariant of the preceding example, we have

$$I_D^m(U) = L_0 x + L_1 y,$$

a linear covariant, the order of L_0 and L_1 being n. It is here assumed that L_0 and L_1 are not identically zero, as they are for the cubic.

3. Every quantic of an odd degree has an invariant of the fourth order in the coefficients.

The discriminant of $I(x, y)^2$ is the required invariant.

4. Every quantic of odd degree n has a seminvariant of the third order in the coefficients which is the leader of a covariant of the n^{th} degree.

For, differentiating with regard to a_n the discriminant obtained in the preceding example, we have, for the resulting seminvariant, $\varpi = 3$, $\kappa = n$, and consequently $\rho = n\varpi - 2\kappa = n$, which is therefore the degree of the covariant of which $\dfrac{d\Delta}{da_n}$ is the leader.

The series of seminvariants obtained in this way for the odd quantics is important, the order in the coefficients being low.

5. When the quintic $(a_0, a_1, a_2, a_3, a_4, a_5)(x, y)^5$ has a triple factor, prove that the covariant I_x is a perfect square, and the covariant J_x a perfect cube, the linear factor being the triple factor of the quintic in both cases.

6. When the quintic has two double factors, the remaining factor is a single factor of J_x.

7. If $U_x = (a_0, a_1, a_2, \ldots a_n)(x, y)^n$, prove that the resultant of U_x and the covariant G_x is the discriminant of U cubed; that is, $R(U_x, G_x) = \Delta^3(U_x)$; and prove also $R(U_x, H_x) = \Delta^2(U_x)$.

8. When the quintic has a triple root, the following symmetric functions of the roots vanish:—

$$\Sigma(\alpha_1 - \alpha_2)^2 \nabla(\alpha_3, \alpha_4, \alpha_5), \quad \Sigma(\alpha_1 - \alpha_2)^4 \nabla(\alpha_3, \alpha_4, \alpha_5).$$

9. Transform two given quadratics in x, y to the forms

$$au^2 + bv^2, \quad a'u^2 + b'v^2,$$

where u and v are linear functions of x and y.

10. If the coefficients of three quadratics

$$a_1x^2 + 2b_1xy + c_1y^2, \quad a_2x^2 + 2b_2xy + c_2y^2, \quad a_3x^2 + 2b_3xy + c_3y^2$$

be connected by the relation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0;$$

prove that they may be reduced by linear transformation to the forms

$$A_1X^2 + C_1Y^2, \quad A_2X^2 + C_2Y^2, \quad A_3X^2 + C_3Y^2.$$

The determinant here written is the condition that the three quadratics should determine a system of points or lines in involution.

11. Prove that two cubics can in general be transformed one into the other by linear transformation.

12. Express three cubics, U, V, W, by means of three cubes.

Assuming

$$\lambda U + \mu V + \nu W \equiv (x - \rho y)^3, \tag{1}$$

and comparing coefficients, we have

$$\lambda a_1 + \mu a_2 + \nu a_3 = 1,$$

$$\lambda b_1 + \mu b_2 + \nu b_3 = -\rho,$$

$$\lambda c_1 + \mu c_2 + \nu c_3 = \rho^2,$$

$$\lambda d_1 + \mu d_2 + \nu d_3 = -\rho^3.$$

These equations, by eliminating λ, μ, ν, give three values of ρ, and corresponding values of λ, μ, ν: in this way we obtain three equations of the form (1) to determine U, V, W in terms of

$$(x - \rho_1 y)^3, \quad (x - \rho_2 y)^3, \quad (x - \rho_3 y)^3.$$

It is easy to see that ρ is given by the equation

$$\begin{vmatrix} a_1\rho + b_1 & a_2\rho + b_2 & a_3\rho + b_3 \\ b_1\rho + c_1 & b_2\rho + c_2 & b_3\rho + c_3 \\ c_1\rho + d_1 & c_2\rho + d_2 & c_3\rho + d_3 \end{vmatrix} = 0.$$

A similar method may be applied to express n quantics of the n^{th} order in terms of n n^{th} powers.

13. Prove that the three roots of a cubic may be expressed as

$$x_1, \quad \theta(x_1), \quad \theta^2(x_1),$$

where

$$\theta(x) = \frac{lx + m}{l'x + m'}, \quad \text{and} \quad \theta^3(x) = x.$$

From Art. 60, putting $\frac{\epsilon}{2}\sqrt{-\frac{\Delta}{3}} = K$, where $\epsilon = 1$ or -1, we derive

$$\begin{aligned} K(\beta - \gamma) &= H\beta\gamma + H_1(\beta + \gamma) + H_2, \\ K(\gamma - \alpha) &= H\gamma\alpha + H_1(\gamma + \alpha) + H_2, \\ K(\alpha - \beta) &= H\alpha\beta + H_1(\alpha + \beta) + H_2. \end{aligned} \qquad (1)$$

These homographic relations between the roots may be written in the form

$$\beta = \theta(\gamma), \quad \gamma = \theta(\alpha), \quad \alpha = \theta(\beta);$$

where the numerator and denominator in θ are supposed to be divided by $2K$; and this being done it will be found that l, m, l', m' are connected by the relations $lm' - l'm = 1 = l + m'$, and the roots α, γ, β may be represented as α, $\theta(\alpha)$, $\theta^2(\alpha)$; $\theta^3(\alpha)$ being equal to α. It is important to observe that the equations (1) are consistent, the sum of the expressions on the right-hand side being zero; that is to say, K must have the same sign in all three, any other combination of signs being inadmissible. (See Serret's *Cours d'Algèbre Supérieure*, vol. ii., art. 511.)

14. Given a binary cubic U and its Hessian H_x, the cubic being satisfied by the ratios $x : y$ and $x' : y'$; prove that

$$F(x, y) \equiv \frac{1}{\sqrt{\Delta}} \, \frac{x'\dfrac{dH_x}{dx} + y'\dfrac{dH_x}{dy}}{xy' - x'y}$$

is an absolute constant, Δ being the discriminant of U.

$F(x, y)$ is absolutely unchanged by linear transformation, since

$$H_{X,Y} = M^2 H_{x,y}, \quad \Delta' = M^6\Delta,$$

and

$$\begin{vmatrix} X & Y \\ X' & Y' \end{vmatrix} = \frac{1}{M}\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}, \qquad X'\frac{d}{dX} + Y'\frac{d}{dY} = x'\frac{d}{dx} + y'\frac{d}{dy}.$$

Reducing U to the sum of two cubes by a linear transformation whose modulus $= 1$, the constant may be easily shown to be $\dfrac{1}{\sqrt{-3}}$. This is another form of the homographic relation of Art. 60.

15. Prove that a rational homographic relation in terms of the coefficients connects any two rational functions of the same root of a cubic equation; but that the relation is not rational when the roots are different.

16. Transform the quartic

$$(a, b, c, d, e)(x, 1)^4$$

into one whose invariant I shall vanish.

Assuming $$y = x^2 + 2\eta x + \zeta,$$

and making the invariant I of the transformed equation vanish, we have

$$\Sigma(\rho_2 - \rho_3)^2(\phi - \rho_1)^2 = 0, \qquad (1)$$

where ϕ is a known quadratic function of η, not involving ζ.

Expanding (1), we have

$$I\phi^2 - 3J\phi + \frac{I^2}{12} = 0,$$

which determines ϕ, and consequently η, by means of a quadratic equation; and ζ may have any value.

By a similar transformation J can be made to vanish.

17. Prove that the most general rational transformation of a quartic $f(x)$ may be reduced to the transformation

$$y = \frac{P}{p-x} + \frac{Q}{q-x}.$$

When $P = Rf(p)\,f'(q)$, and $Q = -Rf(q)\,f'(p)$, show that the second term of the transformed quartic is absent.

18. Prove that the transformation

$$y = \frac{ax^2 + 2\beta x + \gamma}{a_1x^2 + 2\beta_1 x + \gamma_1}$$

may be resolved into the three successive transformations—(1) a homographic transformation; (2) a transformation of the roots into their squares; (3) a homographic transformation.

19. If p be any integer, prove that

$$\frac{(x_1{}^p - x_2{}^p)(x_3{}^p - x_4{}^p)}{(x_1 - x_2)(x_3 - x_4)} = \Sigma_0 + (x_1x_2 + x_3x_4)\,\Sigma_1,$$

where Σ_0 and Σ_1 are symmetric functions of x_1, x_2, x_3, x_4;
and hence prove

$$\frac{(\phi(x_1) - \phi(x_2))(\phi(x_3) - \phi(x_4))}{(\psi(x_1) - \psi(x_2))(\psi(x_3) - \psi(x_4))} = \frac{\Sigma_0 + \Sigma_1(x_1x_2 + x_3x_4)}{\Sigma_0' + \Sigma_1'(x_1x_2 + x_3x_4)},$$

where Σ_0, Σ_1, Σ_0', Σ_1' are symmetric functions of x_1, x_2, x_3, x_4.

20. If $\phi(x, y)$ and $\psi(x, y)$ be two covariants of the binary form

$$U \equiv (a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

of the degrees p and q, respectively; and if

$$\phi\left(xX - \frac{1}{q}\frac{d\psi}{d}Y,\ \ yX + \frac{1}{q}\frac{d\psi}{dx}Y\right)$$

be expanded in the form

$$(V_0, V_1, V_2, \ldots . V_p)(X, Y)^p;$$

prove that $V_0, V_1, V_2, \ldots . V_p$ are covariants of U.

Expanding, the coefficient of $X^{p-j}Y^j$ is

$$\frac{1}{1.2.3\ldots j}\left(\frac{d\psi}{dy}\frac{d}{dx} - \frac{d\psi}{dx}\frac{d}{dy}\right)^j\phi.$$

The modulus of this transformation of ϕ is $\psi(x, y)$.

21. When in the preceding example $n = 4$, and $\phi(x, y)$ and $\psi(x, y)$ are replaced by U, find the values of V_0, V_1, V_2, V_3, V_4.

Ans. $U(1,\ 0,\ H_x,\ G_x,\ IU^2 - 3H^2_x)(X,\ Y)^4$.

22. Prove for two cubics U and V

$$Q^2 = 16 \begin{vmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{vmatrix},$$

where D_{11}, D_{12}, &c., are the invariants of the three Hessians, and Q has the same signification as in Art. 185.

23. Eliminate x' from the equations

$$z = (a_0x' + a_1)x + (a_0x'^2 + 3a_1x' + 2a_2)y, \quad (a_0,\ a_1,\ a_2,\ a_3)(x',\ 1)^3 = 0.$$

Ans. $z^3 + 3H_{x,y}z + G_{x,y} = 0$.

24. Transform the quadric $(a, b, c, f, g, h)(x, y, z)^2$ to X, Y, Z, where

$$X = \alpha_1x + \beta_1y + \gamma_1z, \quad Y = \alpha_2x + \beta_2y + \gamma_2z, \quad Z = \alpha_3x + \beta_3y + \gamma_3z.$$

Ans. $$\begin{vmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & X \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & Y \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & Z \\ X & Y & Z & 0 \end{vmatrix},$$

where $\Pi_{ij} = A\alpha_i\alpha_j + B\beta_i\beta_j + C\gamma_i\gamma_j + F(\beta_i\gamma_j + \beta_j\gamma_i) + G(\gamma_i\alpha_j + \alpha_i\gamma_j) + H(\alpha_i\beta_j + \alpha_j\beta_i)$.

25. Prove that all quartic covariants of U_x whose roots are rational functions of the roots of U_x are included in the formula

$$(\rho^4 + \tfrac{1}{2}I\rho^2 - 2J\rho + \tfrac{1}{16}I^2)U_x - (4\rho^3 - I\rho + J)H_x.$$

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26. When U_x is a quartic, and H_x its Hessian, prove that the factors of $U_xH_y - U_yH_x$ are $x - y$, and the three quadratic factors of G_x (Art. 176) when xy replaces x^2, and $x + y$ replaces $2x$.

27. Prove that $\dfrac{x-\alpha}{\delta-\alpha} + \dfrac{x-\beta}{\delta-\beta} + \dfrac{x-\gamma}{\delta-\gamma}$ is a factor of $I^2U_x - 16JH_x$,

where $$U_x \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

28. If U_x and U'_ξ be two quartics which have the same absolute invariant, prove that

$$IJ'H_xU'_\xi - I'JH'_\xi U_x$$

may be resolved into four factors of the form

$$A_jx\xi + B_jx + C_j\xi + D_j.$$

Mr. Russell.

29. If the leading coefficients of a covariant involve the coefficients of several quantics in the orders $\varpi_1, \varpi_2, \ldots \varpi_r$ and weights $\kappa_1, \kappa_2, \ldots \kappa_r$, the degree of the covariant is

$$n_1\varpi_1 + n_2\varpi_2 + \ldots + n_r\varpi_r - 2(\kappa_1 + \kappa_2 + \ldots + \kappa_r).$$

30. If for every difference $\alpha_p - \alpha_q$, in the formation of a seminvariant ϕ of an equation $U = 0$, we substitute

$$U\frac{(\alpha_p - \alpha_q)}{(x - \alpha_p)(x - \alpha_q)},$$

prove that the result is the product of the covariant whose leader is ϕ by $U^{\kappa-\varpi}$, where ϖ is the order and κ the weight of ϕ.

31. When U is a quintic, what are the invariants of the quartic emanant

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 U?$$

Ans. The quadric and cubic covariants I_x and J_x.

32. Give the relation connecting the covariants H_x, G_x, I_x, J_x, of any quantic U.

Ans. $-G_x^2 = 4H_x^3 - U^2H_xI_x + U^3J_x$.

33. Show how to transform a quantic of an odd order so that all the new coefficients shall be invariants.

Ans. Take two linear covariants for the new X and Y.

34. Find the relation which connects the coefficients of two quartics when their roots are connected by the relation

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = 0.$$

Ans. $I^3J'^2 - I'^3J^2 = 0$.

(See Exs. 13, 14, p. 279; and 13, 14, p. 119.)

SECTION IV.—GEOMETRICAL TRANSFORMATIONS.*

201. **Transformation of Binary to Ternary Forms.**—We think it desirable, before closing the present chapter, to give a brief account of a simple transformation from a binary to a ternary system of variables, whereby a geometrical interpretation may be given to several of the results contained in the preceding chapters. The applications which follow in connexion with the quadratic and quartic will be sufficient to explain this mode of transformation; and will enable the student acquainted with the principles of analytic geometry to trace further the analogy which exists between the two systems.

Denoting the original variables, i.e. the variables of the binary system, by x_0, y_0, we propose to transform to a ternary system by the substitutions

$$x = x_0^2, \quad y = 2x_0y_0, \quad z = y_0^2.$$

For example, taking the simple case of a quadratic whose roots are α, β, viz.,

$$x_0^2 - (\alpha + \beta)\,x_0y_0 + \alpha\beta y_0^2 = 0,$$

and transforming, we obtain

$$x - \tfrac{1}{2}\,(\alpha + \beta)\,y + \alpha\beta z = 0. \qquad (1)$$

We have also the equation

$$y^2 - 4zx = 0.$$

This is the equation of a conic, which we call Γ, and (1) is plainly the equation of a chord of this conic joining the *points* α and β, the point determined by the equations

$$\frac{x}{\phi^2} = \frac{y}{2\phi} = z, \quad \text{where } \phi = \frac{x_0}{y_0},$$

being referred to as the point ϕ on the conic Γ.

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* See *Quarterly Journal of Mathematics*, vol. x., p. 211.

When $a = \beta$ the quadratic becomes $(x_0 - ay_0)^2$, i.e. the square of a factor of the first degree; also (1) reduces to $x - ay + a^2z = 0$, which is plainly the equation of the tangent at the point a to the conic V; whence the line corresponding to a quadratic with distinct roots is a chord of the conic V, this line becoming a tangent when the roots are equal.

In further illustration of this method we consider the binary sextic and quintic, so as to show how the transformation is presented differently according as the degree of the quantic is even or odd. In the former case we have

$$U_0 \equiv (x_0 - a_1y_0)(x_0 - a_2y_0)(x_0 - a_3y_0)(x_0 - a_4y_0)(x_0 - a_5y_0)(x_0 - a_6y_0),$$

which becomes by transformation

$$c_{12}c_{34}c_{56}, \quad c_{12}c_{35}c_{46}, \quad c_{12}c_{36}c_{45},$$

or some other of the fifteen similar products of chords, where $c_{12} \equiv x - \frac{1}{2}(a_1 + a_2)\,y + a_1a_2z$ is the chord 1, 2; and c_{34}, c_{56}, &c., have a like signification. In the second case, viz. when the degree of the binary quantic is odd, we must square U_0 before making the transformation. Thus, if U_0 represents the product of the first five factors written above, U_0^2 becomes when transformed $t_1t_2t_3t_4t_5$, where $t_1 \equiv x - a_1y + a_1^2z$ is the tangent to V at the point a_1, and t_2, t_3, &c., have a like signification.

205. **The Quadratic and Systems of Quadratics.**—The only invariant that a quadratic has is its discriminant, and this is also an invariant in the ternary system, its vanishing being the condition that the line corresponding to the quadratic should touch the conic V. We now consider the system of two quadratics

$$ax_0^2 + 2bx_0y_0 + cy_0^2, \quad a'x_0^2 + 2b'x_0y_0 + c'y_0^2,$$

which for shortness we call L and M.

When transformed these become two lines

$$L \equiv ax + by + cz, \quad M \equiv a'x + b'y + c'z.$$

Now the condition that the line whose equation is $\lambda L + \mu M = 0$ should touch the conic V is

$$\lambda^2 (ac - b^2) + \lambda\mu (ac' + a'c - 2bb') + \mu^2 (a'c' - b'^2) = 0. \qquad (2)$$

All the coefficients of this equation are invariants in both systems: we have already seen that this is true of the first and last coefficients, and the intermediate coefficient which is the harmonic invariant of the binary system is an invariant in the ternary system also, its vanishing expressing the condition that the lines L, M should be conjugate with regard to the conic V. This equation determines the tangents which can be drawn through the point of intersection of L and M to the conic V. When this point is on the conic the tangents coincide, and the discriminant of the quadratic vanishes. Whence we obtain geometrically the following form for the resultant of two quadratics:—

$$R = 4\,(ac - b^2)(a'c' - b'^2) - (ac' + a'c - 2bb')^2;$$

for if L, M, and V have a common point, the original quadratics must have a common root, and the condition is in each case the same.

Again, the pairs of points or lines given by the equation $\lambda L + \mu M = 0$ form a system in involution (cf. Art. 183), the double points or lines being determined by the equation (2); and in the ternary system the corresponding pencil of lines passing through a fixed point determines on a conic a system of points in involution, the double points being the points of contact of tangents drawn to the conic from the fixed point.

If we consider next the three quadratics
$a_1x_0^2 + 2b_1x_0y_0 + c_1y_0^2$, $a_2x_0^2 + 2b_2x_0y_0 + c_2y_0^2$, $a_3x_0^2 + 2b_3x_0y_0 + c_3y_0^2$,
it is seen that the determinant $(a_1b_2c_3)$ is an invariant in both systems, its vanishing being the condition in the binary system that the quadratics should form an involution (Ex. 10, p. 450), and in the ternary system that the three corresponding lines should meet in a point.

As a final illustration, we consider a system of three quadratics connected in pairs by the harmonic relations

$$a_1c_2 + a_2c_1 - 2b_1b_2 = 0, \text{ \&c.}$$

Transforming the quadratics, we obtain three lines X, Y, Z, which form a self-conjugate triangle with regard to the conic V. The theorem relating to three mutually harmonic quadratics, viz. that their squares are connected by an identical linear relation (see Ex. 6, p. 389), is suggested by a well-known property of conics; for V expressed in terms of X, Y, Z is of the form

$$V \equiv X^2 + Y^2 + Z^2;$$

whence, restoring the original variables x_0, y_0, V_0 vanishes identically, and X, Y, Z become the original quadratics, each divided by a factor which may be seen to be the square root of its discriminant (see (1), Ex. 6, p. 389).

206.—**The Quartic and its Covariants treated geometrically**.—It will appear from the remarks to be made in the next Articles that in applying the transformation now under consideration to the quartic $U_0 \equiv (a, b, c, d, e)(x_0, y_0)^4$, the term $6cx_0^2y_0^2$ will be replaced by $2cxz + cy^2$, so that the quartic will be replaced by the two following conics:—

$$U \equiv ax^2 + cy^2 + ez^2 + 2dyz + 2czx + 2bxy,$$

$$V \equiv y^2 - 4zx;$$

the form of U here selected being connected with V by an invariant relation. The invariants of U and V are invariants of the original binary form, for the discriminant of $U - \rho V$ is

$$4\rho^3 - I\rho + J,$$

and the invariants of the ternary system are

$$\Delta' = -4, \quad \Theta' = 0, \quad \Theta = I, \quad \Delta = J;$$

where I and J are the invariants of the quartic, the discriminant of $U - \rho V$ being written as usual under the form

$$\Delta - \rho\Theta + \rho^2\Theta' - \rho^3\Delta'.$$

Let the conics U and V intersect in the points A, B, C, D; these points being determined by the equations

$$\frac{x}{\phi^2} = \frac{y}{2\phi} = z,$$

when ϕ has the four values α, β, γ, δ, the roots of the binary quartic; and let the points of intersection of the common chords BC, AD; CA, BD; AB, CD be E, F, G, respectively, where EFG is the triangle self-conjugate with regard to both conics. Now, denoting by $(\alpha\beta) = 0$ the equation of the line AB, and using a similar notation for the remaining chords, we have by the theory of conics

$$U - \rho_1 V = (\beta\gamma)(\alpha\delta), \quad U - \rho_2 V = (\gamma\alpha)(\beta\delta), \quad U - \rho_3 V = (\alpha\beta)(\gamma\delta),$$

where ρ_1, ρ_2, ρ_3 are the roots of the equation $4\rho^3 - I\rho + J = 0$.

On restoring the original variables x_0, y_0 in these equations, V_0 vanishes identically, and we have U_0 resolved into a pair of quadratic factors in three different ways, depending on the solution of the reducing cubic of the quartic. Whence it appears that the resolution of a quartic into its pairs of quadratic factors, and the determination of the pairs of lines which pass through the four intersections of two conics, are identical problems, each depending on the solution of the same cubic equation.

We now proceed to show that the sides of the common self-conjugate triangle of U, V correspond to the quadratic factors of the sextic covariant in the binary system. Since the side FG is the polar of E, the co-ordinates x', y' of E are found by solving the equations $(\beta\gamma) = 0$, $(\alpha\delta) = 0$; we have, therefore,

$$\frac{x'}{\beta\gamma(\alpha + \delta) - \alpha\delta(\beta + \gamma)} = \frac{y'}{2(\beta\gamma - \alpha\delta)} = \frac{z'}{\beta + \gamma - \alpha - \delta},$$

and, substituting for x', y', z' the values thus determined in the polar of E, viz.,

$$xz' - \frac{yy'}{2} + x'z = 0,$$

we express this equation in the form

$$(\beta + \gamma - \alpha - \delta)x - (\beta\gamma - \alpha\delta)y + (\beta\gamma(\alpha + \delta) - \alpha\delta(\beta + \gamma))z = 0.$$

On restoring the original variables x_0, y_0, this is seen to be one of the quadratic factors of the sextic covariant (Art. 176). It is therefore proved that the points where FG meets V are determined by the quadratic equation

$$(\beta+\gamma-a-\delta)\,\phi^2-2\,(\beta\gamma-a\delta)\,\phi+\beta\gamma\,(a+\delta)-a\delta\,(\beta+\gamma)=0\,;$$

and consequently the six points on V which correspond to the roots of the sextic covariant are the points where this conic meets the sides of the common self-conjugate triangle of U and V.

To determine the points on V which correspond to the roots of the Hessian, we calculate for the conics U and V the covariant conic $\boldsymbol{F}$ (Salmon's *Conic Sections*, Art. 378); thus finding

$$-\tfrac{1}{4}\,\boldsymbol{F}\equiv(ac-b^2)\,x^2+(bd-c^2)\,y^2+(ce-d^2)\,z^2+(be-cd)\,yz$$
$$+\,(ae-2bd+c^2)\,zx+(ad-bc)\,xy\,;$$

and on restoring the original variables, we have

$$H\,(x_0,\,y_0)^4=-\tfrac{1}{4}\,\boldsymbol{F}_0\,;$$

also, since the conic $\boldsymbol{F}$ intersects U and V in the points of contact of their common tangents, we see that the points on V corresponding to the roots of the Hessian are the points so determined. The Hessian has, moreover, a double geometric origin, for it may equally well be obtained by transforming the conic Φ (Salmon's *Conics*, Art. 377) which is the envelope of a line cut harmonically by the conics U and V.

207. We now give some general transformations from the binary system to the ternary, which will be useful in comparing the concomitants in both systems.

(1). *Linear transformation of both systems.*

If the binary variables be linearly transformed, the new variables expressed in terms of the old being

$$X_0=\lambda x_0+\mu y_0,\quad Y_0=\lambda' x_0+\mu' y_0,$$

the new ternary variables will be expressed in terms of the old as follows:—

$$X = \lambda^2 x + \lambda\mu y + \mu^2 z,$$

$$Y = 2\lambda\lambda' x + (\lambda\mu' + \lambda'\mu)\, y + 2\mu\mu' z,$$

$$Z = \lambda'^2 x + \lambda'\mu' y + \mu'^2 z;$$

and, consequently,

$$Y^2 - 4ZX = (\lambda\mu' - \lambda'\mu)^2\,(y^2 - 4zx),$$

showing that *the form of the fixed conic is unaltered* by the above linear transformation of x, y, z, which conversely leads to the general linear transformation of the primitive binary variables. The modulus of this ternary transformation is $(\lambda\mu' - \lambda'\mu)^3$ (see Ex. 4, p. 363).

(2). *Transformation of Partial Differential Coefficients.*

If $f(x_0, y_0)$ becomes U by the substitution of Art. 204, we have

$$\frac{df}{dx_0} = 2x_0\frac{dU}{dx} + 2y_0\frac{dU}{dy},$$

and therefore

$$\frac{d^2f}{dx_0^2} = 2\frac{dU}{dx} + 4\left(x\frac{d^2U}{dx^2} + y\frac{d^2U}{dxdy} + z\frac{d^2U}{dxdz}\right) - 4z\left(\frac{d^2U}{dxdz} - \frac{d^2U}{dy^2}\right),$$

$$= 4\frac{d}{dx}\left(x\frac{dU}{dx} + y\frac{dU}{dy} + z\frac{dU}{dz}\right) - 2\frac{dU}{dx} - 4z\Pi(U),$$

where Π is used to denote the operation $\dfrac{d^2}{dxdz} - \dfrac{d^2}{dy^2}$.

Hence, the degree of f being n, and therefore of U being $\frac{1}{2}n$, we have

$$\frac{d^2f}{dx_0^2} = 2(n-1)\frac{dU}{dx} - 4z\Pi(U), \quad \text{and similarly}$$

$$\frac{d^2f}{dx_0\,dy_0} = 2(n-1)\frac{dU}{dy} + 2y\Pi(U),$$

$$\frac{d^2f}{dy_0^2} = 2(n-1)\frac{dU}{dz} - 4x\Pi(U).$$

If the transformation be such that $\Pi(U)$ vanishes identically, we have, for the transformation of the second differential coefficients, the following simple values:—

$$\frac{d^2f}{dx_0^2} = 2(n-1)\frac{dU}{dx}, \quad \frac{d^2f}{dx_0\,dy_0} = 2(n-1)\frac{dU}{dy}, \quad \frac{d^2f}{dy_0^2} = 2(n-1)\frac{dU}{dz}.$$

From these values we find easily

$$\frac{1}{2}\left(x_0'\frac{d}{dx_0} + y_0'\frac{d}{dy_0}\right)^2 f = (n-1)\left(x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz}\right),$$

showing that *the second emanant* (Art. 168) *in the binary system is transformed into the first polar in the ternary system;* and in like manner all the even emanants are transformed into polar curves of one-half the degree.

Again, if the second differential coefficients of f when expressed in the ternary system be represented as follows:—

$$\frac{d^2f}{dx_0^2} = \phi_1(x, y, z), \quad \frac{d^2f}{dx_0\,dy_0} = \phi_2(x, y, z), \quad \frac{d^2f}{dy_0^2} = \phi_3(x, y, z),$$

we will have

$\Pi(x\phi_1 + y\phi_2 + z\phi_3) \equiv 0$, when $\Pi(\phi_1) \equiv 0$, $\Pi(\phi_2) \equiv 0$, and $\Pi(\phi_3) \equiv 0$;

for $$\frac{d^2\phi_1}{dy_0^2} = \frac{d^2\phi_2}{dx_0\,dy_0} = \frac{d^2\phi_3}{dx_0^2},$$

and therefore by what precedes

$$\frac{d\phi_1}{dz} = \frac{d\phi_2}{dy} = \frac{d\phi_3}{dx};$$

but $$\Pi(x\phi_1 + y\phi_2 + z\phi_3) = \frac{d\phi_1}{dz} + \frac{d\phi_3}{dx} - 2\frac{d\phi_2}{dy},$$

and consequently vanishes identically.

It may be noticed that when $\Pi(\phi_1)$, $\Pi(\phi_2)$, $\Pi(\phi_3)$ do not vanish, we have in general

$$(n-3)\,\Pi(x\phi_1 + y\phi_2 + z\phi_3) = (n-1)\{x\Pi(\phi_1) + y\Pi(\phi_2) + z\Pi(\phi_3)\}.$$

(3). *Transformation of the Jacobian.*

The Jacobian of any binary system u, v is transformed into the Jacobian of U, V, and the fixed conic $4zx - y^2 = W$; W being here used for the conic V of Art. 204, with sign changed, and U, V being the transformed values of u, v. For

$$J(u, v) = \begin{vmatrix} \dfrac{du}{dx_0} & \dfrac{du}{dy_0} \\ \dfrac{dv}{dx_0} & \dfrac{dv}{dy_0} \end{vmatrix} = \frac{1}{(n-1)(n'-1)} \begin{vmatrix} ax_0 + by_0 & bx_0 + cy_0 \\ a'x_0 + b'y_0 & b'x_0 + c'y_0 \end{vmatrix},$$

n and n' being the degrees of u and v, respectively, and a, b, c being used to denote the second differential coefficients; whence we have

$$J(u, v) = \frac{1}{(n-1)(n'-1)} \begin{vmatrix} a & b & c \\ a' & b' & c' \\ y_0^2 & -x_0 y_0 & x_0^2 \end{vmatrix} = \begin{vmatrix} \dfrac{dU}{dx} & \dfrac{dU}{dy} & \dfrac{dU}{dz} \\ \dfrac{dV}{dx} & \dfrac{dV}{dy} & \dfrac{dV}{dz} \\ \dfrac{dW}{dx} & \dfrac{dW}{dy} & \dfrac{dW}{dz} \end{vmatrix},$$

the last determinant being obtained from the preceding by the transformation in (2).

In connexion with this transformation it may be noticed that

$$J(U + \phi W,\ V + \psi W, W) = J(U, V, W) + WJ(\phi, \psi, W);$$

whence it follows that $J(U + \phi W,\ V + \psi W, W)$ and $J(U, V, W)$ give when transformed the same covariant in the binary system.

(4). *The Hessian and other concomitants.*

For the transformation of the Hessian we have

$$n^2(n-1)^2 H(u) = \frac{d^2u}{dx_0^2}\frac{d^2u}{dy_0^2} - \left(\frac{d^2u}{dx_0 dy_0}\right)$$

$$= 4(n-1)^2 \left\{ \frac{dU}{dx}\frac{dU}{dz} - \left(\frac{dU}{dy}\right)^2 \right\},$$

which proves that one curve into which the Hessian may be

transformed is the locus of the poles with regard to U of tangents to the fixed conic.

The line corresponding to the binary concomitant

$$(x_0 y_0' - x_0' y_0)^2 \quad \text{is} \quad xz' - \tfrac{1}{2} yy' + zx',$$

which is the polar of x', y', z' with regard to the fixed conic.

The curve corresponding to the covariant

$$\frac{d^2u}{dx_0^2}\frac{d^2v}{dy_0^2} + \frac{d^2u}{dy_0^2}\frac{d^2v}{dx_0^2} - 2\,\frac{d^2u}{dx_0 dy_0}\,\frac{d^2v}{dx_0 dy_0}$$

is
$$\frac{dU}{dx}\frac{dV}{dz} + \frac{dU}{dz}\frac{dV}{dx} - 2\,\frac{dU}{dy}\frac{dV}{dy},$$

which equated to zero is the condition that the polar lines of a point with regard to U and V should be conjugate with regard to the fixed conic. This covariant may be written under the form $\Pi(UV)$, when $\Pi(U) \equiv 0$, $\Pi(V) \equiv 0$.

208. When the transformation of Art. 204 is applied to a quantic $f(x_0, y_0)$ of even degree $2m$, it is plain that the roots of this quantic will be determined geometrically by the points of intersection of a curve of the m^{th} degree with the fixed conic V. If the degree of the quantic is odd, it must, as already stated, be squared before the transformation is effected; and the roots will then be determined geometrically by the points of contact of the corresponding curve with the conic.

In transforming the quantic $f(x_0, y_0)$, we may obtain an indefinite number of ternary forms by varying the mode of transformation; for if U be any one of these forms, $U + \phi_{m-2} V$, in which the coefficients of ϕ_{m-2} are arbitrary, would equally well be a transformation of $f(x_0, y_0)$, since this form would on restoring the original variables return to the quantic $f(x_0, y_0)$. Moreover, every possible transformation is included in the foregoing. Among these innumerable ternary forms there is always one, such that the invariants and covariants of this form combined with V are invariants and covariants of the binary quantic also. To determine this form take the operator Π of the preceding Article, which, as can be easily seen, is

obtained by substituting the differential symbols D_x, D_y, D_z in the tangential form of V, or D_z, $-2D_y$, D_x for x, y, z in V itself. Operating then with Π on $U + \phi_{m-2}V$, we obtain a result Ψ_{m-2} of the degree $m-2$; and equating to zero its coefficients, we have equations sufficient to determine all the coefficients of ϕ_{m-2}. The required transformation therefore is unique, as these equations are of the first degree.

This mode of fixing the form of $U + \phi_{m-2}V$ is unaltered by any linear transformation of the binary variables and the consequent linear transformation of the ternary variables; for, referring to (1), Art. 207, it is easily proved that the differential operator

$$\frac{d^2}{dZdX} - \frac{d^2}{dY^2} = (\lambda\mu' - \lambda'\mu)^2\left(\frac{d^2}{dzdx} - \frac{d^2}{dy^2}\right);$$

and if after linear transformation any function $f(x, y, z)$ becomes $F(X, Y, Z)$, we have

$$\frac{d^2F}{dZdX} - \frac{d^2F}{dY^2} = (\lambda\mu' - \lambda'\mu)^2\left(\frac{d^2f}{dzdx} - \frac{d^2f}{dy^2}\right),$$

which proves that the form $F(X, Y, Z)$ is fixed by the same law as $f(x, y, z)$, and this law is independent of the linear transformation of the binary system.

The following method may be employed to obtain the proper form of U corresponding to a given binary quantic of even degree. Let the quartic $u = (a_0, a_1, a_2, a_3, a_4)(x_0, y_0)^4$ be written in the form

$$\frac{1}{3\cdot 4}\left\{x_0^2\frac{d^2u}{dx_0^2} + 2x_0y_0\frac{d^2u}{dx_0dy_0} + y_0^2\frac{d^2u}{dy_0^2}\right\};$$

transforming the second differential coefficients, and multiplying the terms by x, y, z, respectively, we obtain the proper form for U, such that $\Pi(U) = 0$, viz.

$$a_0x^2 + a_2y^2 + a_4z^2 + 2a_3yz + 2a_2zx + 2a_1xy.$$

Again, in the case of the sextic u, writing it in the form

$$\frac{1}{5\cdot 6}\left\{x_0^2\frac{d^2u}{dx_0^2}+2x_0y_0\frac{d^2u}{dx_0dy_0}+y_0^2\frac{d^2u}{dy_0^2}\right\};$$

transforming the quartics $\dfrac{d^2u}{dx_0^2}$, $\dfrac{d^2u}{dx_0dy_0}$, $\dfrac{d^2u}{dy_0^2}$ in the manner just explained, and multiplying by x, y, z, respectively, we obtain a ternary cubic U of the proper form (see (2), Art. 207). In a similar manner the transformation of the octavic is made to depend on that of the sextic; and proceeding in this way step by step we may transform any binary quantic u of even degree to a ternary quantic U of half the degree, such that $\Pi(U)\equiv 0$.

209. **Combined System of a Quartic and Quadratic.**—Transforming this binary system we have a ternary system composed of two conics and a line; and for simplicity we shall suppose the conics referred to their common self-conjugate triangle. Denoting the quartic and quadratic by U_0 and l_0, respectively, and the corresponding ternary forms by U and l, we have

$$\begin{aligned} U &= ax^2+by^2+cz^2, & a+b+c&=0,\\ V &= x^2+y^2+z^2, & bc+ca+ab&=I_2,\\ l &= \alpha x+\beta y+\gamma z, & abc&=I_3. \end{aligned}$$

To obtain the linear covariants of this system, since α, β, γ are the co-ordinates of the pole of l with regard to V, the polar of this point with regard to U is $a\alpha x+b\beta y+c\gamma z\equiv m$, the first covariant; and treating m in the same way, $a\alpha$, $b\beta$, $c\gamma$ being the co-ordinates of its pole with regard to V, the polar of this point with regard to U is $a^2\alpha x+b^2\beta y+c^2\gamma z\equiv n$, which is a second covariant. We cannot derive any more linear covariants in this way, for the next one so derived is

$$a^3\alpha x+b^3\beta y+c^3\gamma z=a(bc-I_2)\,\alpha x+b(ca-I_2)\,\beta y+c(ab-I_2)\,\gamma z,$$

and can therefore be expressed in terms of l and m in the form I_3l-I_2m. But three more linear covariants l', m', n', may

be obtained by taking the poles of l, m, n with regard to V, and joining them two and two. This system may be expressed by the Jacobians

$$J(m, n, V), \quad J(n, l, V), \quad J(l, m, V).$$

We have therefore obtained six linear covariants l, m, n, and l', m', n'; to which all others may be reduced, for example

$$\begin{aligned} t_n &= a^n ax + b^n \beta y + c^n \gamma z \\ &= a^{n-2}(bc - I_2)\, ax + b^{n-2}(ca - I_2)\beta y + c^{n-2}(ab - I_2)\,\gamma z \\ &= I_3 t_{n-3} - I_2 t_{n-2}\,; \end{aligned}$$

also

$$b^2c^2ax + c^2a^2\beta y + a^2b^2\gamma z = I_2{}^2 l + I_3 m + I_2 n,$$

since

$$bc = a^2 + I_2, \;\; ca = b^2 + I_2, \;\; ab = c^2 + I_2.$$

Similarly, $b^n c^n ax + c^n a^n \beta y + a^n b^n \gamma z$ may be reduced to the form $Al + Bm + Cn$; and other reductions which present themselves impose no difficulty.

These six linear covariants when transformed give six quadratic covariants in the binary system.

There are six invariants, but only three are special invariants of this system. To obtain them, let the condition that $\lambda l + \mu m + \nu n$ should touch V be

$$D_0\lambda^2 + D_2\mu^2 + D_4\nu^2 + 2D_3\mu\nu + 2D_2\nu\lambda + 2D_1\lambda\mu = 0\,;$$

whence we obtain five invariants, D_0, D_1, D_2, D_3, D_4, where $D_n = a^n\alpha^2 + b^n\beta^2 + c^n\gamma^2$, three of which only are independent, for

$$\begin{aligned} D_n &= a^{n-2}(bc - I_2)\,\alpha^2 + b^{n-2}(ca - I_2)\,\beta^2 + c^{n-2}(ab - I_2)\,\gamma^2 \\ &= I_3 D_{n-3} - I_2 D_{n-2}\,; \end{aligned}$$

whence

$$D_3 = I_3 D_0 - I_2 D_1, \quad D_4 = I_3 D_1 - I_2 D_2\,;$$

and thus we obtain no more than the five invariants I_2, I_3, D_0, D_1, D_2, the two last being special invariants. D_1 vanishes when l and m are conjugate with regard to V, and D_2 when l and n are conjugate with regard to V.

The remaining special invariant may be obtained as the eliminant of l, m, n, viz.

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a\alpha & b\beta & c\gamma \\ a^2\alpha & b^2\beta & c^2\gamma \end{vmatrix} \equiv R_{123}.$$

The square of the last invariant can be expressed in terms of D_0, D_1, D_2, for

$$R^2_{123} = \begin{vmatrix} \alpha & \beta & \gamma \\ a\alpha & b\beta & c\gamma \\ a^2\alpha & b^2\beta & c^2\gamma \end{vmatrix}^2 = \begin{vmatrix} D_0 & D_1 & D_2 \\ D_1 & D_2 & D_3 \\ D_2 & D_3 & D_4 \end{vmatrix};$$

also $$D_3 = I_3D_0 - I_2D_1, \quad D_4 = I_3D_1 - I_2D_2.$$

R_{123} plainly vanishes when l passes through one of the vertices of the common self-conjugate triangle of U and V.

We proceed now to express the resultant of the quadratic and quartic in terms of D_0, D_1, D_2. This is the same problem as to find the condition that l should pass through one of the four points U, V, and is most easily solved by finding the condition that only one conic of the system $U - \rho V$ can be drawn to touch l. Now if l touch $U - \rho V$

$$\rho^2(\alpha^2 + \beta^2 + \gamma^2) - \rho(a\alpha^2 + b\beta^2 + c\gamma^2) + bc\alpha^2 + ca\beta^2 + ab\gamma^2 = 0,$$

or $$D_0\rho^2 - D_1\rho + D_2 + I_2D_0 = 0,$$

and the discriminant of this quadratic is R, whence

$$R \equiv D_1^2 - 4D_0D_2 - 4I_2D_0^2.$$

The geometrical meaning of the relation $D_1 = 0$ is that the line l is cut harmonically by the conics U and V.

To determine the quartic covariants of the binary system from the quadric covariants of the ternary system, we have in the ternary system three quadric covariants, viz. the Jacobians

$$J(l, U, V), \quad J(m, U, V), J(n, U, V);$$

there are also the three conics

$$J(l,\ V,\ W),\quad J(m,\ V,\ W),\quad J(n,\ V,\ W),$$

where $W \equiv a^2x^2 + b^2y^2 + c^2z^2$, the harmonic conic of $ax^2 + by^2 + cz^2$ and $x + y^2 + z^2$ with sign changed.

These three conics are easily reduced, for

$$J(m,\ U,\ V) = J(l,\ V,\ W),\quad J(n,\ U,\ V) = J(m,\ V,\ W);$$

$$J(n,\ V,\ W) = I_2 J(m,\ U,\ V) - I_3 J(l,\ U,\ V);$$

whence there are only three special quadric covariants, and consequently only three special quartic covariants of the binary system.

Before concluding this Article we give some of the forms which would have been obtained if we had employed the ordinary equations of the conics U and V, viz.

$$U \equiv ax^2 + cy^2 + ez^2 + 2dyz + 2czx + 2bxy,$$

$$V \equiv y^2 - 4zx.$$

The condition that $l \equiv \alpha x + \beta y + \gamma z$ should touch $U - \rho V$ is now

$$\Sigma - \rho\Phi + \rho^2\Sigma', \qquad \text{where}$$

$$\Sigma = (ce - d^2)\,\alpha^2 + (ae - c^2)\,\beta^2 + (ac - b^2)\,\gamma^2$$
$$+ 2\,(bc - ad)\,\beta\gamma + 2\,(bd - c^2)\,\gamma\alpha + 2\,(cd - be)\,\alpha\beta,$$

$$\Phi = e\alpha^2 + 4c\beta^2 + a\gamma^2 - 4b\beta\gamma + 2c\gamma\alpha - 4d\alpha\beta,$$

$$\Sigma' = 4\,(\gamma\alpha - \beta^2).$$

Also R_{123} is the Jacobian of Σ, Φ, Σ', considered as conics; and

$$I_2 = -4I,\quad I_3 = -4J,$$

where I and J are as usual the invariants of the quartic.

Examples.

1. If a quartic have a double factor, prove geometrically that this factor is a double factor of H_x.

2. If a quartic have a square factor, prove geometrically that this factor is a quintuple factor of the covariant G_x; and construct the point on the conic V which corresponds to the remaining root of the equation $G_x = 0$.

3. Resolve the quartic as in Art. 179 by finding the tangents to the conic V where U meets it, U and V having been expressed as sums of squares.

4. Determine the condition that $\lambda u + \nu w$ should have two square factors, where u and w are quartics.

Transforming, we have in this case

$$\lambda U + \mu V + \nu W \equiv (\alpha x + \beta y + \gamma z)^2;$$

consequently, every term in the tangential form of $\lambda U + \mu V + \nu W$ must vanish, giving six equations to eliminate $\lambda^2, \mu^2, \nu^2, \mu\nu, \nu\lambda, \lambda\mu$; hence the required condition is determined.

5. Apply the method of transformation of Art. 204 to prove the theorem of Art. 191.

Let Tschirnhausen's transformation be put under the form

$$z = \frac{\alpha x^2 + 2\beta x + \gamma}{\alpha' x^2 + 2\beta' x + \gamma'}. \tag{1}$$

Make the numerator and denominator of the last fraction homogeneous in x, y; replace z by $-\lambda$, and transform: (1) becomes then

$$L + \lambda L' = 0,$$

where
$$L = \alpha x + \beta y + \gamma z, \quad L' = \alpha' x + \beta' y + \gamma' z.$$

If x, y, z be eliminated from the equations $L + \lambda L' = 0$, $U = 0$, $V = 0$, we shall have the transformed quartic in λ; which, considered geometrically, determines the lines drawn from the point of intersection P of L and L' to the points of intersection A, B, C, D of U and V. Again, if κ be so determined that the conic $U + \kappa V$ pass through the point P, the anharmonic ratio of the lines PA, PB, PC, PD, is equal to the anharmonic ratio of the lines TA, AB, AC, AD, where TA is the tangent to $U + \kappa V$ at A; that is, of the lines

$$t + \kappa t', \quad t + \rho_1 t', \quad t + \rho_2 t', \quad t + \rho_3 t',$$

where t and t' are the tangents to U and V at A. Now, forming the invariants of the quartic whose roots are κ, ρ_1, ρ_2, ρ_3, the theorem follows by Arts. 180 and 199.

6. Let three points a, b, c be taken on the conic V given by the equations

$$\rho x = a_1\phi^2 + b_1\phi + c_1, \qquad \rho y = a_2\phi^2 + b_2\phi + c_2, \qquad \rho z = a_3\phi^2 + b_3\phi + c_3,$$

the values of ϕ at these points being α, β, γ, the roots of a cubic U; prove the following constructions for determining the points on the conic corresponding to the roots of the cubic covariant G_x and the Hessian H_x:—

1°. Let tangents be drawn to the conic V at the points a, b, c, forming a triangle ABC, the lines Aa, Bb, Cc meet the conic at points a', b', c' corresponding to the roots of G_x.

2°. The four triangles abc, $a'b'c'$, ABC, $A'B'C'$ are homologous, and their axis of homology meets the conic V at the points corresponding to the roots of H_x.

7. From the constructions in the last example prove that U_x and G_x have the same Hessian H_x, and that the roots of H_x are imaginary when the roots of U_x are real.

—*Dublin Exam. Papers, Bishop Law's Prize*, 1879.

8. Express in terms of their invariants the resultant of the quartic and biquadratic

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2.$$

9. Determine the condition that two quadratic factors $(x-\alpha)(x-\beta)$, $(x-\gamma)(x-\delta)$ of a quartic U_0 should form with a given quadratic $\lambda x^2 + 2\mu x + \nu$ a system in involution.

Transforming, the three corresponding lines must meet in a point, which point is one of the vertices of the common self-conjugate triangle of the conics U and V. The tangential equation of these points is $J(\Sigma, \Sigma', \Phi) = 0$, which is therefore the required condition, the tangential form of $\kappa U + V$ being $\kappa^2\Sigma + \kappa\Phi + \Sigma'$.

This condition may also be put under the form

$$\left(\lambda\frac{d^2}{dy^2} - 2\mu\frac{d^2}{dx\,dy} + \nu\frac{d^2}{dx^2}\right)^3 G_x = 0.$$

10. Prove that the quartics

$$(\alpha_1x^2 + 2\beta_1xy + \gamma_1y^2)(\alpha_3x^2 + 2\beta_3xy + \gamma_3y^2) - (\alpha_2x^2 + 2\beta_2xy + \gamma_2y^2)^2, \qquad (1)$$

$$(\alpha_1x^2 + 2\alpha_2xy + \alpha_3y^2)(\gamma_1x^2 + 2\gamma_2xy + \gamma_3y^2) - (\beta_1x^2 + 2\beta_2xy + \beta_3y^2)^2 \qquad (2)$$

have the same invariants.

Transforming (2) to the ternary system, we have the conic

$$(\alpha_1x + \alpha_2y + \alpha_3z)(\gamma_1x + \gamma_2y + \gamma_3z) - (\beta_1x + \beta_2y + \beta_3z)^2,$$

which for shortness we write as $LN - M^2$, where

$$L = \alpha_1x + \alpha_2y + \alpha_3z, \quad M = \beta_1x + \beta_2y + \beta_3z, \quad N = \gamma_1x + \gamma_2y + \gamma_3z. \qquad (3)$$

Now when the discriminant of

$$LN - M^2 + \lambda (y^2 - 4zx)$$

is formed, the invariants of (2) are the functions $-3H$ and G of this cubic in λ (or the last two coefficients when the second term is removed). This discriminant may be obtained as the resultant of the three equations

$$\begin{aligned} N\alpha_1 - 2M\beta_1 + L\gamma_1 - 4\lambda z &= 0, \\ N\alpha_2 - 2M\beta_2 + L\gamma_2 + 2\lambda y &= 0, \\ N\alpha_3 - 2M\beta_3 + L\gamma_3 - 4\lambda x &= 0, \end{aligned} \qquad (4)$$

when x, y, z are eliminated ; or by eliminating the six quantities x, y, z, L, M, N by means of the three additional equations (3) the resultant is obtained in the form

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & -4\lambda \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 2\lambda & 0 \\ \alpha_3 & \beta_3 & \gamma_3 & -4\lambda & 0 & 0 \\ 0 & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & -\frac{1}{2} & 0 & \beta_1 & \beta_2 & \beta_3 \\ 1 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \equiv \Delta(\lambda).$$

If we had operated similarly on the quartic (1) we should have obtained the same resultant $\Delta(\lambda)$, the form the determinant takes in this case being obtained by dividing the first three rows of $\Delta(\lambda)$ by -4λ, and multiplying the first three columns by -4λ. Whence it follows that the invariants are the same in both cases.

To expand $\Delta(\lambda)$ we replace L, M, N by their values in equations (4), and then eliminate x, y, z, thus obtaining

$$\begin{vmatrix} I_{11} & I_{12} & I_{13} - 2\lambda \\ I_{12} & I_{22} + \lambda & I_{23} \\ I_{13} - 2\lambda & I_{23} & I_{33} \end{vmatrix}, \text{ where } 2I_{pq} = \alpha_p\gamma_q + \alpha_q\gamma_p - 2\beta_p\beta_q.$$

This determinant becomes when expanded

$$4\lambda^3 + 4(I_{22} - I_{13})\lambda^2 - \{I_{11}I_{33} - I_{13}^2 + 4(I_{13}I_{22} - I_{12}I_{23})\}\lambda - \begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{vmatrix},$$

every coefficient of which is the same for both quartics, as may be verified directly.

11. Determine the condition that three quadratics should by linear tranformation be reducible to the forms

$$\frac{d^2\phi}{dx^2}, \quad \frac{d^2\phi}{dxdy}, \quad \frac{d^2\phi}{dy^2}.$$

Ans. $I_{11}I_{33} - 4I_{12}I_{23} + I_{22}^2 + 2I_{22}I_{31} = 0.$

12. Prove that the condition in Ex. 11 is the same for the following two sets of quadratics :—

$$\alpha_1x^2 + 2\beta_1xy + \gamma_1y^2, \quad \alpha_2x^2 + 2\beta_2xy + \gamma_2y^2, \quad \alpha_3x^2 + 2\beta_3xy + \gamma_3y^2,$$

and

$$\alpha_1x^2 + 2\alpha_2xy + \alpha_3y^2, \quad \beta_1x^2 + 2\beta_2xy + \beta_3y^2, \quad \gamma_1x^2 + 2\gamma_2xy + \gamma_3y^2.$$

210. **Principal Concomitants of the Sextic.**—The binary sextic u being the next even form, we shall as a final illustration briefly indicate how its invariants and the two *principal* covariants may be derived from the ternary system of a cubic and conic combined. The two covariants alluded to are the quartic I_{x_0}, whose leader is $a_0a_4 - 4a_1a_3 + 3a_2^2 = I$ and the quadratic $L_{x_0} = I_D(u)$; for by treating these as a combined system, in the manner of Art. 209, we may obtain all the forms of the binary sextic as far as the fourth degree.

Transforming the sextic $u = (a_0, a_1, a_2, a_3, a_4, a_5, a_6)(x_0, y_0)^6$ we have the ternary cubic

$$U \equiv a_0x^3 + a_3y^3 + a_6z^3 + 6a_3xyz$$
$$+ 3\{a_1x^2y + a_2x^2z + a_2y^2x + a_4y^2z + a_4z^2x + a_5z^2y\}.$$

Now forming the discriminant of

$$\frac{1}{6}\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)^2 U - \lambda V',$$

or
$$(U_{11}, U_{22}, U_{33}, U_{23}, U_{31}, U_{12})(x', y', z')^2 - \lambda V',$$

we have
$$4\lambda^3 - I(U)\lambda + J(U), \quad \text{where}$$

$$I(U) = U_{11}U_{33} - 4U_{12}U_{23} + 3U_{22}^2,$$

$$J(U) = \begin{vmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{vmatrix}.$$

Expanding $I(U)$ in the form $(a_{11}, a_{22}, a_{33}, a_{23}, a_{31}, a_{12})(x, y, z)^2$, we find

$$a_{11} = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad 2a_{23} = a_1a_6 - 3a_2a_5 + 2a_3a_4,$$
$$a_{22} = a_1a_5 - 4a_2a_4 + 3a_3^2, \quad 2a_{31} = a_0a_6 - 4a_1a_5 + 7a_2a_4 - 4a_3^2,$$
$$a_{33} = a_2a_6 - 4a_3a_5 + 3a_4^2, \quad 2a_{12} = a_0a_5 - 3a_1a_4 + 2a_2a_3;$$

also

$$J(U) = \begin{vmatrix} a_0x + a_1y + a_2z, & a_1x + a_2y + a_3z, & a_2x + a_3y + a_4z \\ a_1x + a_2y + a_3z, & a_2x + a_3y + a_4z, & a_3x + a_4y + a_5z \\ a_2x + a_3y + a_4z, & a_3x + a_4y + a_5z, & a_4x + a_5y + a_6z \end{vmatrix}.$$

Operating with Π on $I(U)$ we get

$$I_2 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2,$$

viz. the invariant $(1, 2)^6\ U_1U_2$ of the sextic.

Also $\Pi J(U) \equiv L_x$ becomes L_{x_0} on transformation.

Again, if we form the discriminant of

$$I(U) + \tfrac{1}{6}I_2\ V - kV,$$

we have

$$4k^3 - I_4k + I_6,$$

where I_4 and I_6, the invariants of I_{x_0}, are invariants of the fourth and sixth orders of the sextic, the general form of all such invariants being

$$lI_4 + mI_2^2, \quad lI_2I_4 + mI_2^3 + nI_6.$$

The invariants which Dr. Salmon (*Higher Algebra*, p. 262) selects as fundamental are the invariants $-S$ and T of the cubic curve U (*Higher Plane Curves*, Arts. 220, 221).

The condition that the cubic and conic should touch is expressed by the vanishing of an invariant I_{10}, and this invariant is the discriminant of the sextic.

The condition that three connectors of the six points of intersection of U and V should meet in a point is expressed by the vanishing of an invariant I_{15}; this is the skew invariant of the sextic, and may be obtained as the invariant R_{123} of Art. 209 for the combined system

$$I(U) + \tfrac{1}{6}I_2V, \quad V, \quad \Pi J(U).$$

The covariant I_{x_0} may also be obtained from the curve $U_1U_3 - U_2^2$, which transforms into H_{x_0}; for, reducing by the relation $U_{21} = U_{22}$, we find

$$\tfrac{1}{4}\Pi\,(U_1U_3 - U_2^2) = U_{11}U_{33} - 4U_{12}U_{23} + 3U_{22}^2 = I(U).$$

The covariant L_x may also be obtained by substituting $D_z, -2D_y, D_x$ for x, y, z in $I(U) + \frac{1}{6}I_2V$, and operating on U.

NOTES.

NOTE A.

ALGEBRAIC SOLUTION OF EQUATIONS.

The solution of the quadratic equation was known to the Arabians, and is found in the works of Mohammed Ben Musa and other writers published in the ninth century. In a treatise on Algebra by Omar Alkhayyami, which belongs probably to the middle of the eleventh century, is found a classification of cubic equations, with methods of geometrical construction; but no attempt at a general solution. The study of Algebra was introduced into Italy from the Arabian writers by Leonardo of Pisa early in the thirteenth century; and for a long period the Italians were the chief cultivators of the science. A work, styled *L'Arte Maggiore*, by Lucas Paciolus (known as Lucas de Burgo), was published in 1494. This writer adopts the Arabic classification of cubic equations, and pronounces their solution to be as impossible in the existing state of the science as the quadrature of the circle. At the same time he signalizes this solution as the problem to which the attention of mathematicians should be next directed in the development of the science. The solution of the equation $x^3 + mx = n$ was effected by Scipio Ferreo; but nothing more is known of his discovery than that he imparted it to his pupil Florido in the year 1505. The attention of Tartaglia was directed to the problem in the year 1530, in consequence of a question proposed to him by Colla, whose solution depended on that of a cubic of the form $x^3 + px^2 = q$. Florido, learning that Tartaglia had obtained a solution of this equation, proclaimed his own knowledge of the solution of the form $x^3 + mx = n$. Tartaglia, doubting the truth of his statement, challenged him to a disputation

in the year 1535; and in the meantime himself discovered the solution of Ferreo's form $x^3 + mx = n$. This solution depends on assuming for x an expression $\sqrt[3]{t} - \sqrt[3]{u}$ consisting of the difference of two radicals; and, in fact, constitutes the solution usually known as Cardan's. Tartaglia continued his labours, and discovered rules for the solution of the various forms of cubics included under the classification of the Arabic writers. Cardan, anxious to obtain a knowledge of these rules, applied to Tartaglia in the year 1539; but without success. After many solicitations Tartaglia imparted to him a knowledge of these rules; receiving from him, however, the most solemn and sacred promises of secrecy. Regardless of his promises, Cardan published in 1545 Tartaglia's rules in his great work styled *Ars Magna*. It had been the intention of Tartaglia to publish his rules in a work of his own. He commenced the publication of this work in 1556; but died in 1559, before he had reached the consideration of cubic equations. As his work, therefore, contained no mention of his own rules, these rules came in process of time to be regarded as the discovery of Cardan, and to be called by his name.

The solution of equations of the fourth degree was the next problem to engage the attention of algebraists; and here, as well as in the case of the cubic, the impulse was given by Colla, who proposed to the learned the solution of the equation $x^4 + 6x^2 + 36 = 60x$. Cardan appears to have made attempts to obtain a formula for equations of this kind; but the discovery was reserved for his pupil Ferrari. The method employed by Ferrari was a transformation of such a nature as to make both sides of the equation perfect squares; a new unknown quantity being introduced which is itself determined by an equation of the third degree. It is, in fact, virtually the method of Art. 63. This solution is sometimes ascribed to Bombelli, who published it in his treatise on Algebra, in 1579. The solution known as Simpson's, which was published much later (about 1740), is in no respect essentially different from that of Ferrari. In the year 1637 appeared Descartes' treatise, in which are found many improvements in algebraical science, the chief of which are his recognition of the negative and imaginary roots of equations, and his "Rule of Signs." His expression of the biquadratic as the product of two quadratic factors, although deducible immediately from Ferrari's form, was an important contribution to the study of this quantic. Euler's algebra was published in 1770. His solution of the biquadratic (see Art. 61) is

important, inasmuch as it brings the treatment of this form into harmony with that of the cubic by means of the assumed irrational form of the root. The methods of Descartes and Euler were the result of attempts made to obtain a general algebraic solution of equations. Throughout the eighteenth century many mathematicians occupied themselves with this problem; but their labours were unsuccessful in the case of equations of a degree higher than the fourth.

In the solutions of the cubic and biquadratic obtained by the older analysts we observe two distinct methods in operation: the first, illustrated by the assumptions of Tartaglia and Euler, proceeding from an assumed explicit irrational form of the root; the other, seeking by the aid of a transformation of the given function, to change its factorial character, so as to reduce it to a form readily resolvable. In Art. 55 these two methods are illustrated; together with a third, the conception of which is to be traced to Vandermonde and Lagrange, who published their researches about the same time, in the years 1770 and 1771. The former of these writers was the first to indicate clearly the necessary character of an algebraical solution of any equation, viz. that it must, by the combination of radical signs involved in it, represent any root indifferently when the symmetric functions of the roots are substituted for the functions of the coefficients involved in the formula (see Art. 91). His attempts to construct formulas of this character were successful in the cases of the cubic and biquadratic; but failed in the case of the quintic. Lagrange undertook a review of the labours of his predecessors in the direction of the general solution of equations, and traced all their results to one uniform principle. This principle consists in reducing the solution of the given equation to that of an equation of lower degree, whose roots are linear functions of the roots of the given equation and of the roots of unity. He shows also that the reduction of a quintic cannot be effected in this way, the equation on which its solution depends being of the sixth degree.

All attempts at the solution of equations of the fifth degree having failed, it was natural that mathematicians should inquire whether any such solution was possible at all. Demonstrations have been given by Abel and Wantzel (see Serret's *Cours d' Algèbre supérieure*, Art. 516) of the impossibility of resolving algebraically equations unrestricted in form, of a degree higher than the fourth. A transcendental solution, however, of the quintic has been given by M. Hermite, in a form involving elliptic integrals. Among other

contributions to the discussion of the quintic since the researches of Lagrange, one of leading importance is its expression in a trinomial form by means of the Tschirnhausen transformation (see Art. 194). Tschirnhausen himself had succeeded in the year 1683, by means of the assumption $y = P + Qx + x^2$, in the reduction of the cubic and quartic, and had imagined that a similar process might be applied to the general equation. The reduction of the quintic to the trinomial form was published by Mr. Jerrard in his *Mathematical Researches*, 1832–1835; and has been pronounced by M. Hermite to be the most important advance in the discussion of this quantic since Abel's demonstration of the impossibility of its solution by radicals. In a Paper published by the Rev. Robert Harley in the *Quarterly Journal of Mathematics*, vol. vi., p. 38, it is shown that this reduction had been previously effected, in 1786, by a Swedish mathematician named Bring. Of equal importance with Bring's reduction is Dr. Sylvester's transformation (Art. 198), by means of which the quintic is expressed as the sum of three fifth powers, a form which gives great facility to the treatment of this quantic. Other contributions which have been made in recent years towards the discussion of quantics of the fifth and higher degrees have reference chiefly to the invariants and covariants of these forms. For an account of these researches the student is referred to Clebsch's *Theorie der binären algebraischen Formen*, and to Salmon's *Lessons Introductory to the Modern Higher Algebra.*

There has also grown up in recent years a very wide field of investigation relative to the algebraic solution of equations, known as the "Theory of Substitutions." This theory arose out of the researches of Lagrange before referred to, and has received large additions from the labours of Cauchy, Abel, Galois, and other writers. Many important results have been arrived at by these investigators; but the subject is of too great extent and difficulty to find any place in the present work. The reader desirous of information on this subject is referred to Serret's *Cours d'Algèbre supérieure*, and to the *Traité des Substitutions et des Equations algébriques*, by M. Camille Jordan.

NOTE B.

SOLUTION OF NUMERICAL EQUATIONS.

THE first attempt at a general solution by approximation of numerical equations was published in the year 1600, by Vieta. Cardan had previously applied the rule of "false position" (called by him "regula aurea") to the cubic; but the results obtained by this method were of little value. It occurred to Vieta that a particular numerical root of a given equation might be obtained by a process analogous to the ordinary processes of extraction of square and cube roots; and he inquired in what way these known processes should be modified in order to afford a root of an equation whose coefficients are given numbers. Taking the equation $f(x) = Q$, where Q is a given number, and $f(x)$ a polynomial containing different powers of x, with numerical coefficients, Vieta showed that, by substituting in $f(x)$ a known approximate value of the root, another figure of the root (expressed as a decimal) might be obtained by division. When this value was obtained, a repetition of the process furnished the next figure of the root; and so on. It will be observed that the principle of this method is identical with the main principle involved in the methods of approximation of Newton and Horner (Arts. 100, 101). All that has been added since Vieta's time to this mode of solution of numerical equations is the arrangement of the calculation so as to afford facility and security in the process of evolution of the root. How great has been the improvement in this respect may be judged of by an observation in Montucla's *Histoire des Mathématiques*, vol. i., p. 603, where, speaking of Vieta's mode of approximation, the author regards the calculation (performed by Wallis) of the root of a biquadratic to eleven decimal places as a work of the most extravagant labour. The same calculation can now be conducted with great ease by anyone who has mastered Horner's process explained in the text.

Newton's method of approximation was published in 1669; but before this period the method of Vieta had been employed and simplified by Harriot, Oughtred, Pell, and others. After the period of Newton, Simpson and the Bernoullis occupied themselves with the

same problem. Daniel Bernoulli expressed a root of an equation in the form of a recurring series, and a similar expression was given by Euler; but both these methods of solution have been shown by Lagrange to be in no respect essentially different from Newton's solution (*Traité de la Résolution des Equations numériques*). Up to the period of Lagrange, therefore, there was in existence only one distinct method of approximation to the root of a numerical equation; and this method, as finally perfected by Horner, in 1819, remains at the present time the best practical method yet discovered for this purpose.

Lagrange, in the work above referred to, pointed out the defects in the methods of Vieta and Newton. With reference to the former he observed that it required too many trials; and that it could not be depended on, except when all the terms on the left-hand side of the equation $f(x) = Q$ were positive. As defects in Newton's method he signalized—first, its failure to give a commensurable root in finite terms; secondly, the insecurity of the process which leaves doubtful the exactness of each fresh correction; and lastly, the failure of the method in the case of an equation with roots nearly equal. The problem Lagrange proposed to himself was the following:—"Etant donnée une équation numérique sans aucune notion préalable de la grandeur ni de l'espèce de ses racines, trouver la valeur numérique exacte, s'il est possible, ou aussi approchée qu'on voudra de chacune de ses racines."

Before giving an account of his attempted solution of this problem, it is necessary to review what had been already done in this direction, in addition to the methods of approximation above described. Harriot discovered in 1631 the composition of an equation as a product of factors, and the relations between the roots and coefficients. Vieta had already observed this relation in the case of a cubic; but he failed to draw the conclusion in its generality, as Harriot did. This discovery was important, for it led to the observation that any integral root must be a factor of the absolute term of an equation, and Newton's Method of Divisors for the determination of such roots was a natural result. Attention was next directed towards finding limits of the roots, in order to diminish the labour necessary in applying the method of divisors as well as the methods of approximation previously in existence. Descartes, as already remarked, was the first to recognise the negative and imaginary roots of equations; and the inquiry

commenced by him as to the determination of the number of real and of imaginary roots of any given equation was continued by Newton, Stirling, De Gua, and others.

Lagrange observed that, in order to arrive at a solution of the problem above stated, it was first necessary to determine the number of the real roots of the given equation, and to separate them one from another. For this purpose he proposed to employ the equation whose roots are the squares of the differences of the roots of the given equation. Waring had previously, in 1762, indicated this method of separating the roots; but Lagrange observes (*Equations numériques*, Note iii.), that he was not aware of Waring's researches when he composed his own memoir on this subject. It is evident that when the equation of differences is formed, it is possible, by finding an inferior limit to its positive roots, to obtain a number less than the least difference of the real roots of the given equation. By substituting in succession numbers differing by this quantity, the real roots of the given equation will be separated. When the roots are separated in this way Lagrange proposed to determine each of them by the method of continued fractions explained in the text (Art. 105). This mode of obtaining the roots escapes the objections above stated to Newton's method, inasmuch as the amount of error in each successive approximation is known; and when the root is commensurable the process ceases of itself, and the root is given in a finite form. Lagrange gave methods also of obtaining the imaginary roots of equations, and observed that if the equation had equal roots they could be obtained in the first instance by methods already in existence (see Art. 74).

Theoretically, therefore, Lagrange's solution of the problem which he proposed to himself is perfect. As a practical method, however, it is almost useless. The formation of the equation of differences for equations of even the fourth degree is very laborious, and for equations of higher degrees becomes well-nigh impracticable. Even if the more convenient modes of separating the roots discovered since Lagrange's time be taken in conjunction with the rest of his process, still this process is open to the objection that it gives the root in the form of a continued fraction, and that the labour of obtaining it in this form is greater than the corresponding labour of obtaining it by Horner's process in the form of a decimal. It will be observed also that the latter process, in the perfected form to which Horner

has brought it, is free from all the objections to Newton's method above stated.

Since the period of Lagrange, the most important contributions to the analysis of numerical equations, in addition to Horner's improvement of the method of approximation of Vieta and Newton, are those of Fourier, Budan, and Sturm. The researches of Budan were published in 1807 ; and those of Fourier in 1831, after his death. There is no doubt, however, that Fourier had discovered before the publication of Budan's work the theorem which is ascribed to them conjointly in the text. The researches of Sturm were published in 1835. The methods of separation of the roots proposed by these writers are fully explained in Chapter IX. By a combination of these methods with that of Horner, we have now a solution of Lagrange's problem far simpler than that proposed by Lagrange himself. And it appears impossible to reach much greater simplicity in this direction. In extracting a root of an equation, just as in extracting an ordinary square or cube root, labour cannot be avoided; and Horner's process appears to reduce this labour to a minimum. The separation of the roots also, especially when two or more are nearly equal, must remain a work of more or less labour. This labour may admit of some reduction by the consideration of the functions of the coefficients which play so important a part in the theory of the different quantics. If, for example, the functions H, I, and J, are calculated for a given quartic, it will be possible at once to tell the character of the roots (see Art. 68). Mathematicians may also invent in process of time some mode of calculation applicable to numerical equations analogous to the logarithmic calculation of simple roots. But at the present time the most perfect solution of Lagrange's problem is to be sought in a combination of the methods of Sturm and Horner.

NOTE C.

THE PROPOSITION THAT EVERY EQUATION HAS A ROOT.

It is important to have a clear conception of what is proved, and what it is possible to prove, in connexion with the proposition discussed in Arts. 115, 116. If in the equation $a_0x^n + a_1x^{n-1} + \ldots a_n = 0$ the coefficients $a_0, a_1, \ldots a_n$ are used as mere algebraical symbols without any restriction—that is to say, if they are not restricted to denote *numbers*, either real, or complex numbers of the form treated in Chapter XI., then, with reference to such an equation it is not proved, and there exists no proof, that every equation has a root. The proposition which is capable of proof is that, in the case of any rational integral equation of the n^{th} degree, whose coefficients are all complex (including real) numbers, there exist n complex numbers which satisfy this equation; so that, using the terms *number* and *numerical* in the wide sense of Chapter XI., the proposition under consideration might be more accurately stated in the form—*Every numerical equation of the n^{th} degree has n numerical roots.*

As regards this proposition, there appears little doubt that the most direct and scientific proof is one founded on the treatment of imaginary expressions or complex numbers of the kind considered in Chapter XI. The first idea of the representation of complex numbers by points in a plane is due to Argand, who in 1806 published anonymously in Paris a work entitled *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.* This writer some years later gave an account of his researches in Gergonne's *Annales.* Notwithstanding the publicity thus given by Argand to his new methods, they attracted but little notice, and appear to have been discovered independently several years later by Warren in England and Mourey in France. These ideas were developed by Gauss in his works published in 1831; and by Cauchy, who applied them to the proof of the important theorem of Art. 114. With reference to the proposition now under discussion, the proof which we have given in Art. 116 is a modification of a proof found in Argand's original memoir, and reproduced by Cauchy in his *Exercices d'Analyse.* A proof in many respects similar was given by Mourey.

Before the discovery of the geometrical treatment of complex numbers, several mathematicians occupied themselves with the problem of the nature of the roots of equations. An account of their researches is given by Lagrange in Note IX. of his *Equations numériques.* The inquiries of these investigators, among whom we may mention D'Alembert, Descartes, Euler, Foncenex, and Laplace, referred only to equations with rational coefficients; and the object in view was, assuming the existence of factors of the form $x - \alpha$, $x - \beta$, &c., to show that the roots α, β, &c., were all either real, or imaginary quantities of the type $a + b\sqrt{-1}$; in other words, that the solution of an equation with real numerical coefficients cannot give rise to an imaginary root of any form except the known form $a + b\sqrt{-1}$, in which a and b are real quantities. For the proof of this proposition the method employed in general was to show that, in case of an equation whose degree contained 2 in any power k, the possibility of its having a real quadratic factor might be made to depend on the solution of an equation whose degree contained 2 in the power $k - 1$ only; and by this process to reduce the problem finally to depend on the known principle that every equation of odd degree with real coefficients has a real root. Lagrange's own investigations on this subject, given in Note X. of the work above referred to, related, like those of his predecessors, to equations with rational coefficients, and are founded ultimately on the same principle of the existence of a real root in an equation of odd degree with real coefficients.

As resting on the same basis, viz. the existence of a real root in an equation of odd degree, may be noticed two recently published methods of considering this problem—one by the late Professor Clifford (see his *Mathematical Papers*, p. 20, and *Cambridge Philosophical Society's Proceedings*, II., 1876), and the other by Professor Malet (*Transactions of the Royal Irish Academy*, vol. xxvi., p. 453, 1878). Starting with an equation of the $2m^{th}$ degree, both writers employ Sylvester's dialytic method of elimination to obtain an equation of the degree $m(2m - 1)$ on whose solution the existence of a root of the proposed equation is shown to depend; and since the number $m(2m - 1)$ contains the factor 2 once less often than the number $2m$, the problem is reduced ultimately to depend, as in the methods above mentioned, on the existence of a root in an equation of odd degree. The two equations between which the elimination is supposed to be effected are of the degrees m and $m - 1$; and the only difference between the two

modes of proof consists in the manner of arriving at these equations. In Professor Malet's method they are found by means of a simple transformation of the proposed equation, while Professor Clifford obtains them by equating to zero the coefficients of the remainder when the given polynomial is divided by a real quadratic factor. The forms of these coefficients are given in Ex. 38, p. 306; and it will be readily observed that the elimination of β from the equations obtained by making r_0 and r_1 vanish will furnish an equation in α of the degree $m(2m-1)$.

NOTE D.

DETERMINANTS.

The expressions which form the subject-matter of Chapter XII. were first called "determinants" by Cauchy, this name being adopted by him from the writings of Gauss, who had applied it to certain special classes of these functions, viz. the discriminants of binary and ternary quadratic forms. Although Leibnitz had observed in 1693 the peculiarity of the expressions which arise from the solution of linear equations, no further advance in the subject took place until Cramer, in 1750, was led to the study of such functions in connexion with the analysis of curves. To Cramer is due the rule of signs of Art. 118. During the latter part of the eighteenth century the subject was further enlarged by the labours of Bezout, Laplace, Vandermonde, and Lagrange. In the present century the earliest cultivators of this branch of mathematics were Gauss and Cauchy; the former of whom, in addition to his investigations relative to the discriminants of quadratic forms, proved, for the particular cases of the second and third order, that the product of two determinants is itself a determinant. To Cauchy we are indebted for the first formal treatise on the subject. In his memoir on *Alternate Functions*, published in the *Journal de l'Ecole Polytechnique*, vol. x., he discusses determinants as a particular class of such functions, and proves several important general theorems relating to them. A great impulse was given to the study of these expressions by the writings of Jacobi in Crelle's *Journal*, and by his memoirs published

in 1841. Among more recent mathematicians who have advanced this subject may be mentioned Hermite, Hesse, Joachimsthal, Cayley, Sylvester, and Salmon. There is now no department of mathematics, pure or applied, in which the employment of this calculus is not of great assistance, not only furnishing brevity and elegance in the demonstration of known properties, but even leading to new discoveries in mathematical science. Among recent works which have rendered this subject accessible to students may be mentioned Spottiswoode's *Elementary Theorems relating to Determinants*, London, 1851; Brioschi's *La teorica dei Determinanti*, Pavia, 1854; Baltzer's *Theorie und Anwendung der Determinanten*, Leipzig, 1864; Dostor's *Éléments de la théorie des Déterminants*, Paris, 1877; Scott's *Theory of Determinants*, Cambridge, 1880; and the chapters in Salmon's *Lessons Introductory to the Modern Higher Algebra*, Dublin, 1876. For further information on the history of this subject, as well as on that of Eliminants, Invariants, Covariants, and Linear Transformations, the reader is referred to the notes at the end of the work last mentioned.

NOTE E.

COMBINED FORMS.

We give here, as an appendix to Chap. XVII., an enumeration of the concomitants of two quartics U and V. For this purpose it is convenient to use the notation $(\phi, \psi)^p$ for $(1, 2)^p \phi_1\psi_2$, when the distinction between the variables is removed. In this notation we have sixteen covariants $(U_x, V_x)^p$, $(U_x, H'_x)^p$, $(V_x, H_x)^p$, $(H_x, H'_x)^p$, when p has the four values 1, 2, 3, 4; but of these Sylvester has reduced (H_x, H'_x) and $(H_x, H'_x)^2$, so that only fourteen independent covariants are obtained in this way; we have, however, to add the four covariants (H_x, G'_x), (H'_x, G_x), $(H_x, G'_x)^4$, $(H'_x, G_x)^4$. These are the eighteen special covariants of this system (Gordan, *Math. Ann.* II. 275). To this list are to be added the five forms belonging to each quartic separately, viz. U_x, H_x, G_x, I, J, and V_x, H'_x, G'_x, I', J'. Hence there are in all twenty-eight forms made up as follows:—eight invariants, eight quadric, seven quartic, and five sextic covariants.

The Table which follows gives the number of forms of the combined systems from I., I. to IV., IV. :—

	I.	II.	III.	IV.
I.	3	5	13	20
II.		6	15	18
III.			26	61
IV.				28

NOTE F.

THE QUINTIC AND ITS CONCOMITANTS.

GORDAN fixes the number of independent concomitants as twenty-three, which may be derived as follows:—the first fourteen, viz. four invariants, four linear covariants, three quadratic covariants, and three cubic covariants come from the covariants I_x of the second degree and J_x of the third degree considered as a distinct combined system in the manner of Art. 184 ; one reduction, however, in the number there obtained occurs in this case, for the resultant of I_x and J_x, or $R(I_x, J_x)$, is the same as the discriminant of J_x, or $\Delta(J_x)$, both leading to the same invariant of the twelfth order. In addition to the fourteen thus obtained the remaining concomitants are defined as follows, K_x being used to denote the Hessian of J_x:—

Quartic Covariants : $I_D(H_x) = Q_x$, $J(I_x, Q_x)$;

Quintic Covariants : U_x, $J(U_x, I_x)$, $J(U_x, K_x)$;

Sextic Covariants : H_x, $J(I_x, H_x)$,

Septic Covariant : $J(H_x, J_x)$;

Nonic Covariant : $J(U_x, H_x)$.

The foregoing results are collected in the following Table, where

p signifies the degree in the variables, ϖ the order in the coefficients of the quintic, and ν the number of concomitants of each degree:—

p	ϖ				ν
0	4	8	12	18	4
1	5	7	11	13	4
2	2	6	8		3
3	3	5	9		3
4	4	6			2
5	1	3	7		3
6	2	7			2
7	5				1
9	3				1

Adopting the definitions of the invariants given by Clebsch and Gordan, and implied in the following equation (see Art. 183), the connexion between the four invariants of the quintic is established as follows:—

$$-J^2(I_x, K_x) = I_4K_x^2 - 2I_8I_xK_x + I_{12}I_x^2;$$

also
$$\tfrac{1}{8}I_D(J_x) = L_x \equiv L_0x + L_1y.$$

Now, substituting L_1 and $-L_0$ for x and y in I_x, K_x, and $J(I_x, K_x)$,

we find
$$-I_{18}^2 = F(I_4, I_8, I_{12}),$$

since
$$R(I_x, L_x) = 12I_{12} - 16I_4I_8,$$

$$R(K_x, L_x) = I_8^2 - I_4I_{12}.$$

Thus I_{18} is defined, and its square expressed in terms of the other invariants which are not skew.

NOTE G.

THE SEXTIC AND ITS CONCOMITANTS.

The first sixteen forms come from I_x and L_x treated as a combined system (Art. 209). In this way we obtain all the invariants, quadratic covariants, and quartic covariants. There are in general eighteen forms in the combination of a quartic and quadratic, but in this special case, owing to the nature of the coefficients, the invariant D_1, which is an invariant I_8 of the sextic, is expressible in terms of the invariants I_2, I_4, I_6, in the form $I_8 = pI_4^2 + qI_2I_6$: also the covariant sextic of I_x is reducible to those which occur in the enumeration which follows. It should be noticed that since $n\varpi - 2\kappa$ is even for the sextic, all the forms are even in the variables.

The following is a complete enumeration of the covariants :—

Quadrics : $L_x \equiv I_D(U)$, $M_x \equiv L_D(I_x)$, $N_x \equiv M_D(I_x)$,
$J(L_x, M_x)$, $J(L_x, N_x)$, $J(M_x, N_x)$.
Quartics : I_x, $H(I_x)$, $J(I_x, L_x)$, $J(I_x, M_x)$, $J(I_x, N_x)$.
Sextics : U, J_x, $J(U, L_x)$, $J(U, M_x)$, $J(J_x, L_x)$.
Octavics : H_x, $J(U, I_x)$, $J(H_x, L_x)$.
Decimic : $J(I_x, H_x)$.
Duodecimic : G_x.

These results are collected in the following Table, in which p is the degree of the concomitant, ϖ the order in the coefficients, and ν the number of each kind :—

p	ϖ						ν
0	2	4	6	10	15		5
2	3	5	7	8	10	12	6
4	2	4	5	7	9		5
6	1	3	4	6	6		5
8	2	3	5				3
10	4						1
12	3						1

It will be noticed that there are two covariants of the sixth degree in the variables, and of the sixth order in the coefficients; this is the first instance in which there are two seminvariants of the same order and weight in the binary system (see p. 331).

It may be observed that if the ternary forms of any three of the quadratic covariants be taken as lines of reference, the sextic will be represented by a cubic and conic combined, such that every coefficient in the equation of either curve is an invariant of the sextic.

NOTE H.

DETERMINATION OF THE UNIQUE TERNARY FORM.

The following is the simplest method of finding, for a given binary quantic of degree $2m$, the ternary form U, of degree m, such that $\Pi(U) \equiv 0$.

Let U be written with trinomial coefficients complete in form; for the variables x, y, z substitute x_0^2, $2x_0y_0$, y_0^2, respectively, and arrange the result as a binary form of the $2m^{th}$ degree; this form will then become $\Sigma\,(pa_p + qa_q + ra_r + \text{\&c.})\, x_0^{2m-j}\, y_0^{\,j}$, where a_p, a_q, a_r, &c. &c. are the literal coefficients of the ternary form, and p, q, r, &c. &c. their numerical multipliers. The reduction of U from having $\dfrac{m(m+3)}{2} + 1$ coefficients to a form with only $2m + 1$ distinct coefficients will be accomplished by putting $a_p = a_q = a_r$, &c. in such compound terms. When this change is made it will be found that the differential equation $\dfrac{d^2U}{dzdx} - \dfrac{d^2U}{dy^2} = 0$ is satisfied identically: also that $p + q + r +$ &c. is the proper binomial multiplier in the binary form. When, for example, $m = 4$ we have the following quartic for U:—

$$\begin{aligned} a_0x^4 + a_4y^4 + a_8z^4 &+ 6\,(a_6y^2z^2 + a_4z^2x^2 + a_2x^2y^2) \\ &+ 4\,(a_1x^3y + a_2x^3z + a_3y^3x + a_5y^3z + a_6z^3x + a_7z^3y) \\ &+ 12xyz\,(a_3x + a_4y + a_5z), \end{aligned}$$

which becomes on transformation

$$(a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8)(x_0,\ y_0)^8.$$

All the concomitants of this ternary quartic U and the conic $y^2 - 4zx$ combined, are also concomitants of the binary octavic.

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THE END.

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